On Horizon Molecules and Entropy in Causal Sets

论因果集中的视界分子与熵

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Contents

目录

Abstract

摘要

We review the different proposals and attempts to identify the ”horizon molecules” that would give a kinematical estimation for the black hole entropy in causal set theory. The proposals are presented according to their chronological appearance in scientific literature. The review is neither very technical nor merely descriptive; it is aimed to provide the reader with a lucid introduction to the necessary concepts and mathematical background and give him or her a broad view on the subject, by focusing on the main technical and conceptual issues that summarize the progress made in the last two decades.

我们在本文中回顾了因果集理论中，为确定可用于运动学估算黑洞熵的“视界分子”提出的各类方案与尝试。我们按这些方案在科学文献中发表的时间顺序逐一介绍。本篇综述既不过度 technical 也非单纯描述性综述；它旨在为读者清晰介绍相关必要概念与数学背景，并聚焦总结过去二十年研究进展的核心技术与概念问题，让读者对该领域形成全面认识。

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因果集 · 量子引力 · 黑洞 · 熵 · 视界分子 · 统计几何

Introduction

引言

Although the energy scale at which quantum effects on spacetime are expected to show up is well beyond the range of any foreseeable laboratory-based experiments, the theoretical consequences of quantum mechanics and general relativity have been major reasons for studying quantum gravity and searching for a more fundamental structure of spacetime. Most importantly, the discovery of the close relationship between certain laws of black hole physics and the ordinary laws of thermodynamics, on the one hand, and the discovery of the quantum-induced radiation by black hole (BH), on the other hand, appear to be two major pieces of a puzzle that fit together so perfectly that there can be little doubt that this ”fit” is of deep significance [1-5].

尽管预计时空量子效应会显现的能量标度远超出任何可预见实验室实验的可及范围，但量子力学与广义相对论的理论推论一直是研究量子引力、探索时空更基础结构的主要动因。最重要的是，一方面黑洞物理定律与普通热力学定律之间存在紧密关联，另一方面发现黑洞 (BH) 会产生量子诱导辐射，这两项发现仿佛同一谜题的两大核心板块，二者契合得如此完美，几乎可以肯定这种“契合”具有深刻意义 [1-5]。

Today, well into its fifth decade of the development, this merger remains intellectually stimulating and puzzling at once.

> 时至今日，该领域发展已进入第五十年，这项理论融合至今仍兼具智力启发性与谜团性。

One of the most puzzling aspects is the fact black hole possesses an entropy equal to one-quarter of its horizon area expressed in units of Planck area. And in spite of five decades of intensive research, debates, and genuine advances in different directions, especially within the context of string theory and 2+1 gravity [3,6-9] (see also [10] for loop quantum gravity results), it is fair to say that the physical origin of this entropy and all questions accompanying the thermodynamic of BH are still lacking satisfactory answers, and the debate is far from being settled. In particular, it remains uncertain what ”degrees of freedom” or microstates the entropy refers to, or what unavailable information it quantifies. Moreover, it can be said that a well-accepted criterion to select one approach out of the different approaches to quantum gravity or a fundamental theory of nature is its success in solving black hole thermodynamics puzzles in a satisfactory and general manner, in particular revealing the statistical mechanics behind BH entropy.

> 最令人困惑的结论之一是: 黑洞熵等于以普朗克面积为单位的视界面积的四分之一。尽管经过五十年的深入研究、诸多辩论，且在不同方向取得了实质进展——尤其是在弦理论和 2+1 引力框架下 [3,6-9](圈量子引力的相关结果参见 [10])——但客观而言，这种熵的物理起源，以及伴随黑洞热力学产生的所有问题，依然没有令人满意的答案，相关争论远未尘埃落定。具体来说，目前仍不确定这个熵对应的“自由度”或微观态究竟是什么，也不确定它量化的是哪部分不可获取的信息。此外，可以说，能否以令人满意的通用方式解决黑洞热力学谜题，尤其是揭示黑洞熵背后的统计力学原理，已经成为从不同量子引力方法中筛选正确理论、筛选自然界基础理论的公认标准。

It also is generally believed that all the puzzles of the BH are not independent and will be solved once we really solve one of them. For this and other reasons, providing a controllable calculation of BH entropy has been a prime target of all theories and proposals to quantum gravity.

> 人们还普遍认为，黑洞的所有谜题并非相互独立，只要我们真正解决其中一个，所有谜题都会迎刃而解。出于这一点以及其他原因，对黑洞熵给出可控制的计算，一直是所有量子引力理论和研究方案的核心目标。

Indeed, in the current climate the role being played by BH thermodynamics in this connection looks more and more analogous to the role played historically by the thermodynamics of a box of gas and black body radiation in revealing the underlying atomicity and quantum nature of everyday matter and radiation. This analogy can be brought out more clearly by recalling some facts about thermodynamics in the presence of event horizons.

> 事实上，在当前研究环境下，黑洞热力学在这方面发挥的作用，越来越类似历史上容器气体热力学和黑体辐射在揭示普通物质与辐射的原子性和量子本质过程中所起的作用。我们可以通过回顾事件视界存在情况下热力学的一些基本事实，把这个类比梳理得更清晰。

A well-accepted definition of entropy is as a measure of missing or ”unavailable” information about a physical system, and from this point of view, one would have to expect some amount of entropy to accompany an event horizon, since it is by definition an information hider par excellence, and therefore, the BH entropy could be understood as a response of having an event horizon which hides information about a region of space time, and here the notion of entanglement entropy comes into play. This originates from the well-known observation that an observer outside the horizon has no access to the degrees of freedom behind the horizon. For this reason the outside observer would describe the world with a reduced density matrix obtained by tracing out the inaccessible degrees of freedom behind the horizon. If the exterior modes and the external modes are correlated ”entangled,” the resulting density operator is thermal even if the global state of the system is pure [11,12].

熵被广为接受的定义是: 它衡量物理系统中缺失的或“不可获取”的信息。从这个角度看，事件视界必然会伴随一定的熵，因为事件视界本身就是最典型的信息隐藏者，因此黑洞熵可以理解为事件视界隐藏了时空某一区域信息所带来的结果，而纠缠熵的概念正是在这里发挥作用。这一概念起源于一个广为人知的结论: 视界外的观测者无法获取视界内部的自由度。因此，视界外的观测者需要对不可获取的内部自由度求迹，得到约化密度矩阵来描述世界。如果外部模式和内部模式存在关联——即“纠缠”——那么即使整个系统的全局态是纯态，最终得到的密度算符也是热型的 [11,12]。

Now, what modes or missing information the BH entropy refers to generally remains a mystery. Nevertheless, in the presence of a horizon, in principle one should associate to each quantum field an ”entanglement entropy” that necessarily results from tracing out the interior modes of the field, given that these modes are necessarily correlated with the exterior ones. In the continuum, this entanglement entropy turns out to be infinite, at least when calculated for a free field on a fixed background spacetime. However, if one imposes a short distance cutoff on the field degrees of freedom, one obtains instead a finite entropy, and if the cutoff is chosen around the Planck length, then this entropy has the same order of magnitude as that of the horizon [13, 14]. Based on this appealing result, there have been many speculations attributing the black hole entropy to the sum of all the entanglement entropies of the fields in nature [5]. Whether or not the entanglement of quantum fields furnishes all of the entropy or part of it, contributions of this type must be present, and any consistent theory must provide for them in its thermodynamic accounting.

到目前为止，黑洞熵究竟对应哪些模式、哪些缺失信息，整体上依然是个谜。但在存在视界的情况下，原则上每个量子场都必然对应一个“纠缠熵”: 因为量子场的内部模式必然和外部模式存在关联，对内部模式求迹就一定会得到纠缠熵。在连续场中，至少当我们在固定背景时空下计算自由场的纠缠熵时，结果会是发散的。不过如果我们对场的自由度引入短距离截断，就能得到有限的熵；如果截断长度取在普朗克长度量级，得到的熵就和视界熵处于同一数量级 [13, 14]。基于这个引人注目的结果，已有许多猜想认为黑洞熵就是自然界所有量子场纠缠熵的总和 [5]。无论量子场纠缠贡献了全部还是部分熵，这类贡献必然存在，任何自洽的理论在热力学计算中都必须将其纳入考量。

It is not, of course, the aim of this introduction to give an account of the developments in different directions that have surrounded the entanglement entropy in connection with black holes, and the reader is referred, for instance, to [15] and references therein. However, there is a growing consensus that entanglement entropy, and in general quantum entanglement and holography, will play a central role in revealing a finer structure of spacetime and possibly leading to a radical revision of our perception of the universe.

当然，这篇引言的目的并非梳理与黑洞纠缠熵相关的各方向发展，感兴趣的读者可以参阅 [15] 及其中的参考文献。但目前越来越多人达成共识: 纠缠熵，乃至广义上的量子纠缠与全息原理，将在揭示时空更精细结构方面发挥核心作用，甚至可能彻底革新我们对宇宙的认知。

At present, and without having at hand a viable and more fundamental theory of spacetime, it is hard to expect a resolution of the problem of the divergence of entanglement entropy, which is very likely deeply linked to other issues of BH thermodynamics. Nevertheless, the finiteness of the BH entropy, on the one hand, and the behavior of the entanglement entropy in the continuum picture, on the other hand, seem to point directly towards an underlying discrete structure of spacetime. The situation actually appears to be similar to that of an ordinary box of gas, where we know that, fundamentally, the finiteness of the entropy rests on the finiteness of the number of molecules and to lesser extent on the discreteness of their quantum states. Indeed, at temperatures high enough to avoid quantum degeneracy, the entropy is, up to a logarithmic factor, merely the number of molecules composing the gas. The similarity with the BH becomes evident when we remember that the picture of the horizon as composed of discrete constituents gives a good account of the entropy if we suppose that each such constituent occupies roughly one unit of Planck area and carries roughly one bit of entropy [2].

目前，我们还没有现成的可行且更基础的时空理论，很难解决纠缠熵的发散问题，该问题很可能与黑洞热力学的其他问题深度关联。尽管如此，一方面黑洞熵的有限性，另一方面连续图景中纠缠熵的行为，似乎都直接指向时空具有底层离散结构。实际情况和普通气箱的情形十分相似: 我们知道，从根本上说，熵的有限性依赖于分子数量的有限性，其次才依赖于量子态的离散性。确实，在温度足够高、可避免量子简并的情况下，忽略对数因子后，熵就等于组成气体的分子数。当我们想到，如果假设视界的每个离散组分大致占据一个普朗克面积单位、携带约 1 比特的熵，那么由离散组分构成视界的图景就能很好地解释熵，黑洞情形与气箱的相似性就一目了然了 [2]。

A proper statistical derivation along these lines would require a knowledge of the dynamics of these constituents, of course. However, in analogy with the gas, one may still anticipate that the horizon entropy can be estimated by counting suitable discrete structures, analogs of the gas molecules, without referring directly to their dynamics. Clearly, this type of estimation can succeed only if well-defined discrete entities can be identified which are available to be counted. Within a continuum theory, it is hard to think of such entities. However, in causal set theory [16], the elements of the causal set serve as ”spacetime atoms,” and one can ask whether these elements, or some related structures, are suited to play the role of ”horizon molecules.”

当然，沿着这一思路进行恰当的统计推导，需要先了解这些组分的动力学。不过，和气体的情况类比，我们仍然可以预见，即便不直接用到组分的动力学，也可以通过对合适的离散结构 (即气体分子的类似物) 计数来估算视界熵。显然，只有当能识别出可计数的、定义明确的离散实体时，这种估算才能成立。在连续理论中很难找到这类实体，但在因果集理论 [16] 中，因果集的元素就充当“时空原子”的角色，我们可以探究这些元素或是相关结构是否适合扮演“视界分子”的角色。

The idea of considering a certain causal set structure as a potential candidate for the horizon molecules was first taken up in 1999 using causal links. This proposal was partially successful and gave promising results in two dimensions. It was subsequently followed by other proposals to refine it or look for more suitable definitions for the horizon molecules that would work in higher spacetime dimensions.

将某种因果集结构作为候选视界分子的想法最早在 1999 年通过因果关联展开研究。该提议取得了部分成功，在二维下得到了很有前景的结果。随后又出现了其他提议，对其进行优化，或是为视界分子寻找更适配的定义，使其能在更高维的时空中适用。

In this review, we go through the different horizon molecule proposals that emerged in the last two decades or so within the causal set approach to quantum gravity. The different proposals will be presented according to their chronological appearance in literature. We therefore shall first focus on the causal link proposal that appeared in [17, 18] , which historically was the first proposal and so far seems to be the simplest one, and in spite of the fact that it has turned out to be unsuccessful beyond two dimensions, this proposal remains pedagogically useful and conceptually stimulating. As a consequence of the failure of the link proposal in higher dimensions, other horizon molecule proposals were put forward in subsequent and recent years aiming to succeed where the first proposal failed [19-21]. These subsequent and recent proposals will then be reviewed, and their main results will be reported and discussed.

本篇综述梳理了近二十多年来量子引力因果集方法中，陆续提出的各类视界分子方案。我们会按这些方案在文献中出现的时间顺序逐一介绍: 我们首先关注最早出现在 [17, 18] 的因果关联方案，它从历史上讲就是首个方案，也是目前为止最简单的方案；尽管该方案被发现在二维以上不成立，但在教学上仍有价值，在概念上也具有启发性。由于关联方案在高维下失效，后续近年又出现了其他视界分子方案，目标是解决首个方案未能解决的问题 [19-21]。我们随后也会综述这些近年提出的新方案，汇报并讨论它们的主要结果。

This review is not intended to be a full comprehensive survey on this subject; however, we hope that the material presented herein will offer the beginner researcher in the subject, or the interested theoretical physicist in general, an accessible introduction to the subject, enough background, tools, and concepts that enable him or her to understand the abovementioned efforts and developments to identify the horizon molecules in causal set theory and direct the reader to the still open issues.

本篇综述并非对该课题的全面调研，但我们希望此处呈现的内容，能为该方向的新手研究者或是感兴趣的普通理论物理学家提供入门指引，给出足够的背景知识、工具与概念，帮助其理解在因果集理论中识别视界分子的相关工作与进展，同时指明该领域仍待解决的开放问题。

Background and Terminology

背景与术语

In this section we give the essential mathematical definitions and terminology related to the causal set picture of spacetime. We shall limit ourselves to the necessary background relevant to this review. For more comprehensive and extensive introduction to causal set hypothesis we refer the reader to [22,23]; for a recent and broad review with a fuller set of references, see [24].

本节我们给出与时空因果集合图景相关的基本数学定义和术语，我们仅介绍本综述所需的必要背景。关于因果集合假设更全面详尽的介绍，读者可参阅文献 [22,23]；包含更完整参考文献的最新宽泛综述可参阅 [24]。

Definition 1. A causal set (or a causet for short) $\mathcal{C}$ is a set endowed with an order relation $\prec$ satisfying the following axioms:

定义 1. 一个因果集合 (简称因果集) $\mathcal{C}$ 是一个赋予了序关系 $\prec$ 的集合，该序关系满足以下公理:

1. Acyclic (antisymmetric): $\forall p, q \in \mathcal{C}$, $p \prec q$ and $q \prec p \Rightarrow p = q$.

1. 无圈性 (反对称性): $\forall p, q \in \mathcal{C}$, $p \prec q$ and $q \prec p \Rightarrow p = q$.

2. Transitive: $\forall p, q, r \in \mathcal{C}, p \prec q \prec r \Rightarrow p \prec r$.

2. 传递性: $\forall p, q, r \in \mathcal{C}, p \prec q \prec r \Rightarrow p \prec r$.

3. Reflexive: $\forall p \in \mathcal{C}, p \prec p$.

3. 自反性: $\forall p \in \mathcal{C}, p \prec p$.

4. Locally finite: $\forall p, q \in \mathcal{C}, |I[p,q]| < \infty$, where $I[p,q] = \mathrm{Fut}(p) \cap \mathrm{Past}(q)$, $|.|$ stands for the cardinality of the set, Fut and Past denote the future and the past of a given point,

4. 局部有限性: $\forall p, q \in \mathcal{C}, |I[p,q]| < \infty$, 其中 $I[p,q] = \mathrm{Fut}(p) \cap \mathrm{Past}(q)$, $|.|$ 代表集合的基数，Fut 和 Past 分别表示给定点的未来和过去,

$$\mathrm{Fut}(p) = \{q \in \mathcal{C} \mid p \prec q, q \neq p\}$$

$$\mathrm{Past}(p) = \{q \in \mathcal{C} \mid q \prec p, q \neq p\}.$$

Notice here that the reflexivity axiom is a matter of convention and we could instead have used the irreflexive convention.

请注意，自反公理是约定俗成的，我们也可以采用非自反的约定。

Fut (p) and Past (p) are to be compared with the notion of chronological future and past, $I^+(p)$ and $I^-(p)$, in continuum Lorentzian geometry. $I[p,q]$ is referred to as the causal or order interval, the analogue of Alexandrov interval in the continuum.

未来集 Fut (p) 和过去集 Past (p) 可对应连续洛伦兹几何中的时序未来和时序过去 $I^+(p)$ 和 $I^-(p)$ 。$I[p,q]$ 被称为因果区间或序区间，是连续情况下亚历山德罗夫区间的对应物。

The discreteness of the causal set is encoded in the local finiteness axiom.

因果集合的离散性由局部有限性公理体现。

The acyclicity axiom ensures that causets do not have closed causal loops.

无圈性公理保证因果集不存在闭合因果回路。

An important concept for the description of causets and that we shall frequently need is the Link.

描述因果集时我们经常会用到的一个重要概念是链接。

Definition 2. Let p and $q \in \mathcal{C}, p \prec q, q \neq p$. If $|I[p,q]| = 0$, we say there is a link between p and q and write $p \prec \cdot q$.

定义 2. 设 p and $q \in \mathcal{C}, p \prec q, q \neq p$. 若 $|I[p,q]| = 0$ ，我们称 p 与 q 之间存在链接，记为 $p \prec \cdot q$.

The knowledge of all links is equivalent to knowledge of all relations among elements: $p \prec q$ iff there are elements $q_1, q_2, , \dots \dots q_n$ such that $p \prec \cdot q_1 \prec \cdot q_2 \prec \cdot, \dots . \prec \cdot q_n \prec \cdot q$. Therefore, links are irreducible relations and in some sense are the building blocks of the causet.

知道所有链接就等价于知道所有元素间的关系: $p \prec q$ 当且仅当存在元素 $q_1, q_2, , \dots \dots q_n$ 使得 $p \prec \cdot q_1 \prec \cdot q_2 \prec \cdot, \dots . \prec \cdot q_n \prec \cdot q$. 因此，链接是不可约关系，在某种意义上是因果集的构造单元。

Definition 3. Let $\mathcal{C}' \subset \mathcal{C}, p \in \mathcal{C}'$ which is said to be maximal (resp.minimal) in $\mathcal{C}'$ iff it is in the past (resp. future) of no other element in $\mathcal{C}'$.

定义 3. 设 $\mathcal{C}' \subset \mathcal{C}, p \in \mathcal{C}'$ ，当且仅当它在 $\mathcal{C}'$ 中不位于任何其他元素的过去 (对应极大元)/未来 (对应极小元) 时，称其为 $\mathcal{C}'$ 中的极大元 (对应极小元)。

An extended notion of maximality and minimality condition that will later be needed is the notion of maximal and minimal-but- n .

我们后续会用到一个拓展的极大极小概念，即极大与极小- n 概念。

Definition 4. Let $\mathcal{C}' \subset \mathcal{C}, p \in \mathcal{C}'$ which is said to be maximal-but- n (resp.minimal-but- n) in $\mathcal{C}'$ iff it is in the past (resp. future) of exactly n elements in $\mathcal{C}'$.

定义 4。设 $\mathcal{C}' \subset \mathcal{C}, p \in \mathcal{C}'$ 为 $\mathcal{C}'$ 中的极大但- n (相应地，极小但- n)，当且仅当它恰好位于 $\mathcal{C}'$ 中 n 个元素的过去 (相应地，未来)。

The basic hypothesis of the causal set approach to quantum gravity is that ”spacetime, ultimately, is discrete and its underlying structure is that of a locally finite, partial ordered set which continues to make sense even when the standard geometrical picture ceases to do so.” The macroscopic spacetime continuum we experience must be recovered as an approximation to the causet. The causal set proposal can roughly be summarized in the following two points:

量子引力因果集方法的基本假设是:“时空本质上是离散的，其基础结构是局部有限的偏序集；即便标准几何图像不再适用，该结构仍然成立。”我们所体验的宏观连续时空，必须作为因果集的一种近似被还原出来。因果集的主张大致可归纳为以下两点:

1. Quantum gravity is a quantum theory of causal sets.

1. 量子引力是关于因果集的量子理论。

2. A continuum spacetime $(\mathcal{M}, g)$ is an approximation of an underlying causal set $C \sim (\mathcal{M}, g)$, where

2. 连续时空 $(\mathcal{M}, g)$ 是基础因果集 $C \sim (\mathcal{M}, g)$ 的一种近似，其中

(a) Order $\sim$ causal order

(a) 序对应 $\sim$ 因果序

(b) Number $\sim$ spacetime volume

(b) 数量对应 ~ 时空体积

Point or step (2) is not to be viewed as independent of step (1). Actually the quantum theory of causal set should dictate how the continuum picture emerge as an approximation, and this could ultimately involve a more sophisticated notion of approximation. For instance, in view of the fact that not all causets admit a realization as spacetimes with a given dimension while respecting conditions (2a) and (2b), the process by which the continuum 4-d spacetime picture, or that of higher-dimensional spacetimes with compactified extra dimensions, is reached may involve some sort of coarse-graining in which the manifold picture would be a scale -dependent approximation of the causal set. However, in the absence of a quantum dynamics of causet, a systematic way of defining a coarse-graining that would fit automatically our expectations is yet to be discovered. Nevertheless, we may use point (2) as a stepping stone (given) to investigate possible kinematical consequences of the causet approach. In short and without expanding too much around this point, the intuitive idea at work here is that of a faithful embedding which we define below.

上述第 (2) 点不能看作独立于第 (1) 点。实际上，因果集的量子理论应当指明连续图像是如何作为近似涌现出来的，这最终可能需要一种更复杂的近似概念。举例而言，并非所有因果集都能在满足条件 (2a) 和 (2b) 的前提下实现为给定维度的时空，因此得到连续 4 维时空图景 (或是额外维紧致化的高维时空图景) 的过程可能涉及某种粗粒化，流形图像在此过程中是因果集依赖于尺度的近似。然而，目前还没有因果集的量子动力学，也尚未发现一种系统的方式来定义能自动符合我们预期的粗粒化。尽管如此，我们仍可以将第 (2) 点作为进阶基础 (已知条件) 来研究因果集方法可能得到的运动学结论。简而言之，在此不做过多展开，此处的核心直观思想是我们下文定义的忠实嵌入。

Definition 5. If $(\mathcal{M}, g)$ is a d -dimensional Lorentzian manifold and $\mathcal{C}$ a causet, then a faithful embedding of $\mathcal{C}$ into $\mathcal{M}$ is an injection map $f : \mathcal{C} \hookrightarrow \mathcal{M}$ of the causet into the manifold that satisfies the following requirements:

定义 5。若 $(\mathcal{M}, g)$ 是 d 维洛伦兹流形，$\mathcal{C}$ 是一个因果集，则将 $\mathcal{C}$ 忠实嵌入 $\mathcal{M}$ 是一个满足以下条件的从因果集到流形的单射 $f : \mathcal{C} \hookrightarrow \mathcal{M}$：

1. The causal relations induced by the embedding agree with those of $\mathcal{C}$ itself,

1. 嵌入诱导的因果关系与 $\mathcal{C}$ 自身的因果关系一致，

i.e., $x \prec y \Leftrightarrow f(x) \in J^-(f(y))$

即，$x \prec y \Leftrightarrow f(x) \in J^-(f(y))$

where $J^-(p)$ stands for the causal past of p in $\mathcal{M}$.

其中 $J^-(p)$ 代表 p 在 $\mathcal{M}$ 中的因果过去。

2. The embedded points are distributed uniformly at density $\varrho_c = l_c^{-d}$ with respect to the spacetime volume measure of $(\mathcal{M}, g)$.

2. 相对于 $(\mathcal{M}, g)$ 的时空体积测度，嵌入点以密度 $\varrho_c = l_c^{-d}$ 均匀随机分布。

3. The characteristic length over which the geometry varies appreciably is everywhere much greater than the mean spacing between the embedded points.

3. 几何发生明显变化的特征长度处处远大于嵌入点之间的平均间距。

l_c is referred to as the discreteness scale.

l_c 被称为离散标度。

When these conditions are satisfied, the spacetime $(\mathcal{M}, g)$ is said to be a continuum approximation to $\mathcal{C}$ and we write $\mathcal{C} \sim (\mathcal{M}, g)$.

当这些条件都满足时，我们称时空 $(\mathcal{M}, g)$ 是 $\mathcal{C}$ 的连续近似，记为 $\mathcal{C} \sim (\mathcal{M}, g)$ 。

To ensure covariance the above embedding is realized by randomly sprinkling in points until the required density is reached. Therefore, from the point of view of $\mathcal{M}$ the causet resembles a ”random Lattice,” e.g., ”a regular” lattice cannot do the job since it is not uniform in all frames or coordinate systems.

为保证协变性，上述嵌入通过随机洒布点直到达到要求密度的方式实现。因此，从 $\mathcal{M}$ 的角度来看，因果集类似于“随机格点”；规则格点无法完成这个任务，因为它无法在所有参考系或坐标系中保持均匀。

A natural choice for obtaining or creating a faithfully embedded causet is via a Poisson point process, under which the probability to find n elements in a spacetime region of volume V is given by

得到或构建一个忠实嵌入因果集的自然选择是通过泊松点过程，根据该过程，在体积为 V 的时空区域中找到 n 个元素的概率为

$$(\varrho_c V)^n \frac{e^{-\varrho_c V}}{n!} \tag{1}$$

This makes $f(\mathcal{C})$ a random causet, and thereby, any function $F : \mathcal{C} \to \mathbb{R}$ is a random variable.

这使得 $f(\mathcal{C})$ 成为一个随机因果集，因此，任意函数 $F : \mathcal{C} \to \mathbb{R}$ 都是一个随机变量。

For more detailed discussion of the issue of faithful embedding and the probabilistic nature of the process, we refer the reader to [24] and references therein.

关于忠实嵌入问题和该过程的概率性质的更详细讨论，读者可参考文献 [24] 及其中的参考文献。

Horizon Molecules as Causal Links

视界面分子作为因果连接

As discussed in the introduction, the expectation is that the BH entropy can be understood as entanglement in a sufficiently generalized sense, and we may hope to estimate its leading behavior by counting suitable discrete structures that measure the potential entanglement in some way between in- and outside discrete structures. Moreover, and owing to the fact that the entropy essentially measures the horizon area in Planck units, the problem is reduced to coming up with this measure in the causal set picture.

正如引言中所述，我们预期黑洞熵可以在足够广义的意义上从纠缠的角度理解，并且我们有望通过计数合适的离散结构来估算它的领头阶行为，这些离散结构某种程度上衡量了视界内外离散结构之间潜在的纠缠。此外，由于熵本质上是用普朗克单位衡量视界面面积，这个问题在因果集图像中就简化为构造出这个度量。

It is worthy of note here that it seems far from obvious that such structures must exist. If they do, then they provide a relatively simple order theoretic measure of the area of a cross section of a null surface, and, unlike what one's Euclidean intuition might suggest, it is known that such measures are not easy to come by. For example, no one knows such a measure of spacelike distance between two sprinkled points that works in general, though some progress has been made in such Minkowski spacetime [25].

值得注意的是，这类结构是否必然存在远非显而易见。如果它们确实存在，就能为类光曲面截面的面积提供一个相对简单的序理论度量；而且和欧几里得直觉可能给出的结论不同，目前已知这类度量并不容易得到。例如，尽管在闵氏时空已经取得了一些进展 [25]，但目前还没人能给出一个适用于一般情况的、度量两个撒布点之间类空间隔的这类度量。

It follows from the above discussion that a natural and the simplest candidate for the structure we seek is a link crossing the horizon. Indeed, we may think heuristically of ”information flowing along links” and producing entanglement when it flows across the horizon during the course of the causet's growth (or ”time development”). Since links are irreducible causal relations (in some sense the building blocks of the causet), it seems natural that by counting links between elements that lie outside the horizon and elements that lie inside, one would measure the degree of entanglement between the two regions. Equally, it seems natural that the number of such causal links if supplemented with extra conditions might turn out to be proportional to the horizon area and play the role of the horizon molecules.

从上述讨论可以推出，我们寻找的结构中最自然也最简单的候选就是穿过视界面的连接。实际上，我们可以粗略地认为「信息沿着连接流动」，在因果集生长 (即「时间演化」) 的过程中，信息穿过视界面就产生了纠缠。由于连接是不可约的因果关系 (在某种意义上是因果集的构件)，计数位于视界外的元素和视界内的元素之间的连接，自然就可以衡量两个区域之间的纠缠程度。同样，这类因果连接的数量在补充额外条件后，很自然会和视界面面积成正比，从而扮演视界面分子的角色。

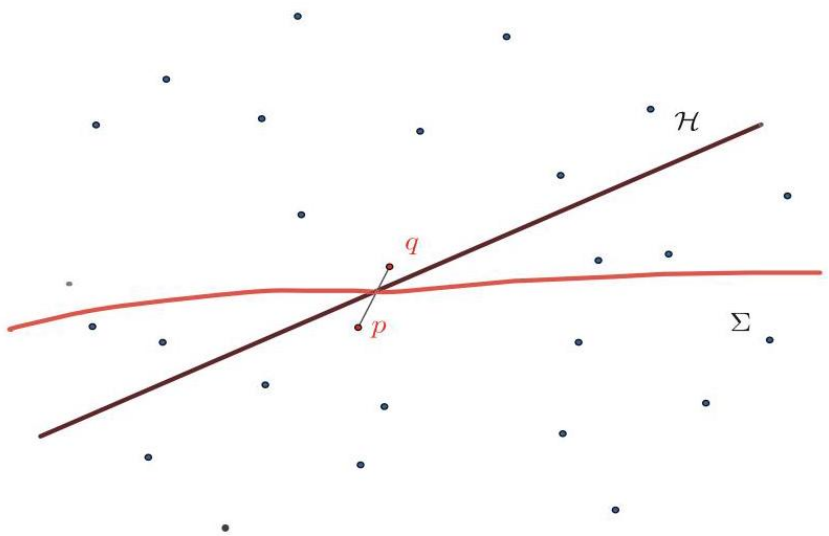


Fig. 1 A typical geometrical setting showing a typical causal link crossing the horizon

图 1 展示典型因果连接穿过视界面的典型几何设置

In what follows we discuss with some detail the link proposal for horizon molecules and its applications in different 1+1 geometrical setups.

下文我们将详细讨论视界面分子的连接提案，及其在不同 1+1 维几何设置中的应用。

The General Setup

一般设置

Let us consider a causet $\mathcal{C}$ obtained via Poisson random sprinkling in a black hole background $\mathcal{M}$ with density ϱ_c , so this causal set is faithfully embeddable in this geometrical background by definition. Let $\mathcal{H}$ be a BH horizon and let $\sum$ be an achronal hypersurface intersecting the horizon, Fig. 1.

我们考虑一个在黑洞背景 $\mathcal{M}$ 中通过泊松随机洒出得到密度为 ϱ_c 的因果集合 $\mathcal{C}$ ，根据定义，该因果集合可忠实嵌入到这个几何背景中。设 $\mathcal{H}$ 为黑洞视界，$\sum$ 为与视界相交的类时超曲面，见图 1。

The goal is to come up with a measure of the area of the resulting cross section between $\mathcal{H}$ and $\sum$, which in turn would measure the horizon entropy and define horizon molecules.

我们的目标是提出一个方法，测量 $\mathcal{H}$ 与 $\sum$ 之间所得截面的面积，进而测量视界熵并定义视界分子。

A natural and intuitive candidate for such molecules is to take them made of pairs of points (p, q) , with p lying outside the black hole and to the past of $\sum$, while q is inside the black hole and to the future of $\sum$, and $p \prec \cdot q$, i.e., $p \prec q$ is a link.

这类分子的一个自然直观的候选方案是，将其构造为点对 (p, q) : 其中 p 位于黑洞外、$\sum$ 的过去，q 位于黑洞内、$\sum$ 的未来，且满足 $p \prec \cdot q$ ，即 $p \prec q$ 是一个因果链接。

If no further conditions are imposed on p and q , the expected number of such links can easily be shown to diverge.

如果不对 p 和 q 施加额外条件，不难证明这类链接的期望数是发散的。

To see what conditions must be imposed on the pairs (p, q) , let us remember that, intuitively, what we are trying to estimate is not the total sum of all ”lost information” but only that corresponding ”to a given time,” meaning in the vicinity of the given hypersurface $\sum$. Hence, to associate the same causal link with more than one hypersurface would be to ”overcount” it in forming our estimate, and it is this overcounting that seems to be the source of the abovementioned divergence.

为了弄清需要对点对 (p, q) 施加什么条件，我们回想一下，直观上我们要估计的不是所有“丢失信息”的总和，而只是对应“给定时刻”的部分，也就是给定超曲面 $\sum$ 附近的部分。因此，如果同一个因果链接被关联到多个超曲面，会在我们的估计中造成“重复计数”，而正是这种重复计数导致了上述发散。

Therefore, further conditions are needed to be imposed to give a definition of the horizon molecules which is truly proper to $\sum$ rather than to some earlier or later hypersurface. Several possibilities suggest themselves for this purpose, but none seems to be clearly best, as the end result (the leading order) will be shown to be insensitive to which choice one makes. Below we pick up a specific choice or definition of horizon molecules, which will be referred to as the ”causal link proposal,” and the general issue will be discussed further in section ”On the Min/Max Conditions”. Actually working out explicitly with this particular choice, seeing its success in 1+1 and failure in higher dimensions, due to IR divergence, will be instructive for the reader to conceive the motivations behind the redefinitions of the horizon molecules that subsequently departed from the original link proposal.

因此，我们需要施加额外条件，给出一个真正属于 $\sum$ 而非属于更早或更晚超曲面的视界分子定义。为此有多种可能的方案，但没有哪一种明显最优，而且我们会看到，领头阶的最终结果对具体选择不敏感。下文我们选取一个特定的视界分子选择/定义，将其称为“因果链接方案”，并在“论最小/最大条件”一节进一步讨论一般性问题。对读者而言，实际使用这个特定方案推导，观察它在 1+1 维的成功和因红外发散在高维的失败，将有助于理解后续偏离原始链接方案重新定义视界分子背后的动机。

The causal link proposal (Dou-Sorkin 1999) A horizon molecule with respect to a given hypersurface $\sum$ is a pair (p, q) satisfying the following conditions:

因果链接方案 (Dou-Sorkin 1999): 相对于给定超曲面 $\sum$ 的一个视界分子，是满足以下条件的点对 (p, q) :

---

1. $p \in I^-(\sum) \cap I^-(\mathcal{H})$.
2. $q \in I^+(\sum) \cap I^+(\mathcal{H})$.

3. $|I[p,q]| = 0$, i.e., $p \prec q$ is a link.
4. p is maximal in $I^-(\sum) \cap I^-(\mathcal{H})$ and q is minimal in $I^+(\mathcal{H})$.

The 4th condition may seem asymmetric, as one would have expected symmetric max and min conditions between p and q to be more natural; however, the reason that we do not impose a similar condition on q is because this would give zero for a null hypersurface case, but the result should agree for null or spacelike if both intersect the horizon in the same time; moreover, for stationary black the results should agree in all cases.

第四个条件看起来不对称——人们本以为 p 和 q 之间对称的最大最小条件会更自然；但我们不对 q 施加类似条件的原因是，那样会在零超曲面情形得到零结果，而如果类零超曲面和类空超曲面同时与视界相交，结果应当一致；此外，对于稳态黑洞，所有情形下的结果都应当一致。

Before we move on, we draw the reader's attention that throughout this section and the next one p will stand for points in $I^-(\sum) \cap I^-(\mathcal{H})$ and q for the ones in $I^+(\sum) \cap I^+(\mathcal{H})$.

在继续推导之前，请读者注意: 在本节和下一节中，p 代表 $I^-(\sum) \cap I^-(\mathcal{H})$ 中的点，q 代表 $I^+(\sum) \cap I^+(\mathcal{H})$ 中的点。

Let us now see how to count the expected number of these horizon molecules by reducing it to the calculation of an integral over the manifold $\mathcal{M}$.

现在我们来看如何通过将其约化为流形 $\mathcal{M}$ 上的积分计算，来统计这些视界分子的期望数。

Remember that the probability of finding or sprinkling n points in some region of spacetime, $\mathcal{R}$, is given by the Poisson distribution

记得在时空某区域找到或洒出 n 个点的概率 $\mathcal{R}$ 服从泊松分布:

$$P(n,\mathcal{R}) = \frac{(\varrho_c \,\mathrm{vol}(\mathcal{R}))^n}{n!} e^{-\varrho_c \,\mathrm{vol}(\mathcal{R})}$$

where $\mathrm{vol}(\mathcal{R})$ is the spacetime volume of $\mathcal{R}$.

其中 $\mathrm{vol}(\mathcal{R})$ 是 $\mathcal{R}$ 的时空体积。

Consider first an infinitesimal region $\Delta\mathcal{R}$; the probability of sprinkling a single point in it follows from

首先考虑无穷小区域 $\Delta\mathcal{R}$ ，洒出单个点的概率可由下式得到:

$$P(1,\Delta\mathcal{R}) \approx \varrho_c \,\mathrm{vol}\,\Delta\mathcal{R} \equiv \varrho_c \Delta V. \tag{2}$$

Consider now two infinitesimal regions $\Delta\mathcal{R}_p \in I^+(\sum) \cap I^+(\mathcal{H})$ and $\Delta\mathcal{R}_q \in I^-(\sum) \cap I^-(\mathcal{H})$. The probability of having a pair of points (p,q) with $p \in I^+(\sum) \cap I^+(\mathcal{H})$ and $q \in I^-(\sum) \cap I^-(\mathcal{H})$ sprinkled in $\Delta\mathcal{R}_p$ and $\Delta\mathcal{R}_q$ resp. is given by

现在考虑两个无穷小区域 $\Delta\mathcal{R}_p \in I^+(\sum) \cap I^+(\mathcal{H})$ 和 $\Delta\mathcal{R}_q \in I^-(\sum) \cap I^-(\mathcal{H})$。分别在 $\Delta\mathcal{R}_p$ 和 $\Delta\mathcal{R}_q$ 中撒入满足 $p \in I^+(\sum) \cap I^+(\mathcal{H})$ 和 $q \in I^-(\sum) \cap I^-(\mathcal{H})$ 的点对 (p,q) 的概率由下式给出

$$P\left(p,q \mid \Delta\mathcal{R}_p, \Delta\mathcal{R}_q\right) = \varrho_c \Delta V_p \varrho_c \Delta V_q. \tag{3}$$

If we further require the relation between p and q to be a link, then the Alexandrov interval $A(p,q)$ between p and q must contain no point, and therefore, the probability becomes

如果我们进一步要求 p 和 q 之间的关系为一个链接，那么 p 与 q 之间的亚历山德罗夫区间 $A(p,q)$ 必须不包含任何点，因此概率变为

$$P\left(p \prec \cdot q \mid \Delta\mathcal{R}_p, \Delta\mathcal{R}_q\right) = P\left(0, A(p,q)\right) \varrho_c^2 \Delta V_p \Delta V_q$$

$$= \varrho_c^2 e^{-\varrho_c \operatorname{vol}(A(p,q))} \Delta V_p \Delta V_q. \tag{4}$$

In addition to the link condition max and min conditions must be imposed on p and q . The max and min conditions are just statements about an extra region in $\mathcal{M}$ being empty, with no sprinkled points. If we denote by $\mathcal{R}(p,q)$ the region resulting from the union of $A(p,q)$, $I^+(p) \cap I^-(\sum) \cap I^-(\mathcal{H})$ and $I^-(q) \cap I^-(\mathcal{H})$, the probability for the above link to become a horizon molecule reduces to

除链接条件外，还必须对 p 和 q 施加最大与最小条件。最大与最小条件本质上就是要求 $\mathcal{M}$ 中的一个额外区域为空，不包含任何撒入的点。若我们用 $\mathcal{R}(p,q)$ 表示 $A(p,q)$, $I^+(p) \cap I^-(\sum) \cap I^-(\mathcal{H})$ 与 $I^-(q) \cap I^-(\mathcal{H})$ 的并集区域，那么上述链接成为视界分子的概率可化简为

$$P\left(\mathbf{H}(p,q); \Delta\mathcal{R}_p, \Delta\mathcal{R}_q\right) = \varrho_c^2 e^{-\varrho_c V(p,q)} \Delta V_p \Delta V_q \tag{5}$$

where $V(p,q) = \operatorname{vol}(\mathcal{R}(p,q))$.

其中 $V(p,q) = \operatorname{vol}(\mathcal{R}(p,q))$。

To count the expected number of horizon molecules, we remember that the existence of a horizon molecule is a random variable generated by a function whose value is 1 if the horizon molecule conditions are fulfilled and 0 otherwise. With this in mind, it follows that the expected number of horizon molecules is obtained by summing in (5) over all $p \in I^-(\sum) \cap I^-(\mathcal{H})$ and $q \in I^+(\sum) \cap I^+(\mathcal{H})$ in the limit ΔV_p and ΔV_q go to zero. In this limit the sums are replaced by integrals over the domain of p and q to obtain the following final expression for the expected number of horizon molecules:

为了计算预期的视界分子数量, 我们注意到, 视界分子的存在是一个随机变量: 当满足视界分子条件时该变量取值为 1，否则为 0。据此，预期的视界分子数量可通过对 (5) 式中所有 $p \in I^-(\sum) \cap I^-(\mathcal{H})$ 和 $q \in I^+(\sum) \cap I^+(\mathcal{H})$ 在 ΔV_p 和 ΔV_q 趋于零的极限下求和得到。在该极限中，求和被替换为对 p 和 q 定义域的积分，最终得到如下预期视界分子数量的表达式:

$$< \mathbf{H}_{\text{link}} > = \varrho_c^2 \int_{I^-(\sum) \cap I^-(\mathcal{H})} dV_p \int_{I^+(\sum) \cap I^+(\mathcal{H})} dV_q e^{-\varrho_c V(p,q)}. \tag{6}$$

For a more systematic derivation of the above integral formula, see [19].

上述积分公式的更系统推导参见文献 [19]。

For horizon molecules as such to be successful, one has to show that in the limit of large density, or l_c is much smaller than the geometrical length scales of the setting, $< \mathbf{H}_{\text{link}} >$ has the asymptotic form

要让这种视界分子的构造成立，必须证明: 在高密度极限下，也就是 l_c 远小于该场景的几何长度标度时，$< \mathbf{H}_{\text{link}} >$ 的渐近形式为

$$\varrho_c^{\frac{d-2}{d}} < \mathbf{H}_{\text{link}} >= a^{(d)} \int_{\mathcal{J}} dV_{\mathcal{J}} + \cdots \tag{7}$$

where the dots refer to terms vanishing in the continuum limit. $\mathcal{J} := \sum \cap \mathcal{H}$ and $dV_{\mathcal{J}}$ is the surface measure on $\mathcal{J}.a^{(d)}$ is constant that depends on the dimension of the spacetime but, in principle, not on the nature of $\sum$, null or spacelike. In two dimensions the leading term in $< \mathbf{H}_{\text{link}} >$ should be just a constant.

其中省略号代表连续极限中趋于零的项。$\mathcal{J} := \sum \cap \mathcal{H}$ 和 $dV_{\mathcal{J}}$ 中 $\mathcal{J}.a^{(d)}$ 上的面测度是一个仅依赖时空维度的常数，原则上不依赖 $\sum$ 是类光还是类空的性质。二维情形下，$< \mathbf{H}_{\text{link}} >$ 的首项应当就是一个常数。

Horizon Molecules and the Area Law in Two Dimensions

二维视界分子与面积律

Ideally, one would have used (6) to evaluate the expected number of horizon molecules, $< \mathbf{H}_{\text{links}} >$, in a full four-dimensional BH background, e.g., Schwarzschild BH; however, historically and for technical reasons (difficulties) a simplified version was first worked out. This consisted in considering a "dimensionally reduced" two-dimensional metric instead of the true four-dimensional one. The hope was twofold; it would first be a warm-up exercise for a more realistic four-dimensional BH; second, the establishment of the area law in 2-d models would give strong evidence for the validity of this proposal in the full four-dimensional case. Stated differently, the four-dimensional answer would differ from the two-dimensional one only by a fixed proportionality coefficient of order one, together with a factor of the horizon area.

理想情况下，人们本可以用式 (6) 计算完整四维黑洞背景 (例如史瓦西黑洞) 中视界分子的期望数 $< \mathbf{H}_{\text{links}} >$；但出于历史原因与技术层面的困难，人们首先推导了简化版本，即考虑一个“维度约化”的二维度规而非真实的四维度规。最初的期望有两点: 首先这是处理更现实的四维黑洞问题的热身练习；其次，在二维模型中建立面积律将为该方案在完整四维情形下的有效性提供有力证据。换言之，四维结果与二维结果仅相差一个量级为 1 的固定比例系数，再乘以一个视界面积因子。

Now, although the above-defined horizon molecule proposal did not work beyond 2-d, in contrast to what had first been hoped, due to IR divergences, the establishment of the area law in 2-d using the above-defined horizon molecules makes the calculation worth discussing. Besides this obvious reason, it will be seen that in 1+1 the resulting expected number of links seems to exhibit some interesting features: a sort of universality, giving exactly the same answer for two different geometrical backgrounds, in equilibrium and far from equilibrium, and remaining finite in the strict continuum limit, $\varrho_c \to \infty$.

如今，尽管上述定义的视界分子方案因红外发散，未能如最初期望那样推广到二维以上，但利用上述定义的视界分子在二维中建立面积律这一工作仍值得讨论。除此之外，我们会发现，在 1+1 维中，得到的联络期望数似乎展现出一些有趣的性质: 它具有某种普适性，对平衡态和远离平衡态的两种不同几何背景给出完全相同的结果，并且在严格连续极限下仍保持有限，$\varrho_c \to \infty$ 。

In the sequel two cases will explicitly be worked out, a 2-d reduced Schwarzschild geometry and collapsing null shell. We shall set $\varrho_c = 1$ in all 2-d models discussed in this section, because the leading term is a dimensionless constant and the subleading ones are easy to express and control in these units.

下文将明确计算两种情形: 二维约化史瓦西几何与坍缩零壳。本节讨论的所有二维模型中我们都取 $\varrho_c = 1$ ，因为领头项是无量纲常数，次领头项在这些单位下也易于表达和控制。

An Equilibrium Black Hole: 2-D Reduced Model

平衡黑洞: 二维约化模型

Consider a dimensionally reduced Schwarzschild spacetime obtained from the realistic 4-dimensional BH spacetime, outside a collapsing spherically symmetric star, by identifying each 2-sphere S^2 to a point. The resulting two-dimensional spacetime has exactly the same causal structure as the S-sector of the 4-dimensional one. The Penrose diagram for this spacetime is depicted in Fig. 2. For simplicity the presence of the collapse has been ignored; this of course will not change the argument, since the detail of the collapse should be irrelevant, or one can choose the hypersurface to intersect the horizon far from the collapse and the result will not be affected by the presence of collapse.

考察从真实四维黑洞时空 (坍缩球对称恒星外部) 中，将每个二维球面 S^2 认同为一个点得到的维度约化史瓦西时空。所得二维时空的因果结构与四维时空的 S 扇区完全一致。该时空的彭罗斯图如图 2 所示。为简化我们忽略了坍缩的存在；这当然不会改变论证，因为坍缩的细节应当是无关的，也可以选择超曲面在远离坍缩处与视界相交，结果不会受到坍缩存在的影响。

The line element of the resulting spacetime is obtained by omitting the angular coordinates from the four-dimensional line element, namely,

所得时空的线元可通过从四维线元中去掉角坐标得到，即:

$$d^2s = -\frac{4a^3}{r}e^{-r/a}dudv \tag{8}$$

where $a = 2M$ is the radius of the BH and u and v are the usual Kruskal-Szekeres coordinates, with r defined implicitly by the equation

其中 $a = 2M$ 是黑洞半径，u 和 v 是常规克鲁斯卡尔-塞凯赖什坐标，r 由以下方程隐式定义:

$$uv = \left(1 - \frac{r}{a}\right) e^{r/a} \tag{9}$$

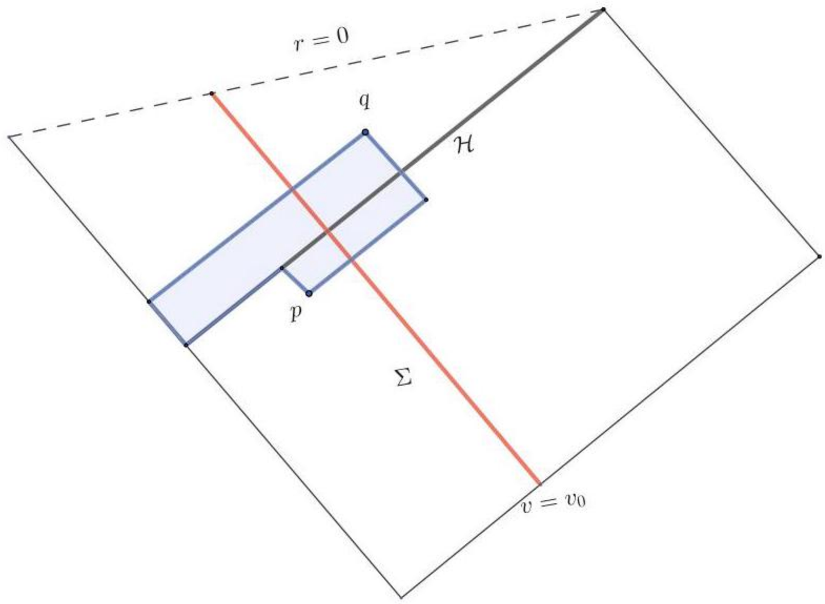


Fig. 2 An equilibrium BH obtained from the real 4-d Schwarzschild BH by dimensional reduction, keeping only the radial section. The shaded region is required to be free from any sprinkled points and with volume $V(p,q)$

图 2 对真实四维史瓦西黑洞做维数约化，仅保留径向截面得到的平衡黑洞。阴影区域要求不存在任何随机撒点，体积为 $V(p,q)$

The associated volume element is

相关体积元为:

$$dV = \sqrt{-g}dudv = \frac{2a^3}{r}e^{-r/a}dudv. \tag{10}$$

Our sign convention is such that $u \sim t - r, v \sim t + r$, and the horizon $\mathcal{H}$ coincides with $u = 0$. Let now $\sum$ be an ingoing null hypersurface defined by the equation $v = v_0$. The shaded region depicted in Fig. 2 is the region $\mathcal{R}(p,q)$ with no sprinkled point, its $V(p,q)$ volume can readily be evaluated using (9)

我们的符号约定为 $u \sim t - r, v \sim t + r$，视界 $\mathcal{H}$ 与 $u = 0$ 重合。现在设 $\sum$ 是由方程 $v = v_0$ 定义的入射类光超曲面。图 2 中阴影区域为无随机撒点的区域 $\mathcal{R}(p,q)$，其 $V(p,q)$ 体积可由 (9) 直接计算:

$$V = a^2 + r_{pq}^2 - r_{pp}^2 - r_{qq}^2 \tag{11}$$

where we have introduced the following notation:

此处我们引入了如下记号:

$$u_i v_j = \left(1 - \frac{r_{ij}}{a}\right) e^{r_{ij}/a}. \tag{12}$$

Let us note that in two dimensions and for a null $\sum$, the maximality condition on p is actually redundant and insured by the link condition, but it would be needed with spacelike $\sum$.

需要注意，在二维情况下，对于类光 $\sum$ ，p 上的极大性条件实际上是多余的，它已由联络条件保证，但类空 $\sum$ 仍需要该条件。

Using (6) and (11), the expected number of horizon molecules is given by

利用 (6) 和 (11)，可得视界分子的期望数目为:

$$< \mathbf{H}_{\text{link}} >= \left(2a^3\right)^2 \int_0^{v_0} dv_p \int_{-\infty}^{0} du_p \int_{v_0}^{\infty} dv_q \int_0^{1/v_q} du_q \frac{e^{-r_{pp}/a - r_{qq}/a}}{r_{pp} r_{qq}} e^{-V}.$$

(13)

A change of integration variables from $\left(u_p, v_p, u_q, v_q\right)$ to $\left(r_{pp}, r\left(u_p, v_0\right) \equiv \ r_{p0}, r_{pq}, r_{qq}\right)$, followed by the notational substitutions $x = r_{pq}, y = r_{p0}, z = r_{pp}$, reduces $< \mathbf{H}_{\text{link}} >$ to the form

将积分变量从 $\left(u_p, v_p, u_q, v_q\right)$ 换为 $\left(r_{pp}, r\left(u_p, v_0\right) \equiv \ r_{p0}, r_{pq}, r_{qq}\right)$, 再做记号替换 $x = r_{pq}, y = r_{p0}, z = r_{pp}$ ，可将 $< \mathbf{H}_{\text{link}} >$ 化简为如下形式:

$$< \mathbf{H}_{\text{link}} >= 4I(a)J(a)$$

where

其中

$$I(a) = \int_a^{\infty} dx \frac{x}{x-a} e^{-x^2} \int_a^x dy \frac{y}{y-a} \int_a^y e^{z^2} dz \tag{14}$$

and

且

$$J(a) = e^{-a^2} \int_0^a e^{r_{qq}^2} dr_{qq}. \tag{15}$$

It is worth noting here that the initial explicit dependence of $< \mathbf{H}_{\text{link}} >$ on v_0 has disappeared, reflecting the stationarity of the black hole.

此处值得注意，$< \mathbf{H}_{\text{link}} >$ 对 v_0 的初始显式依赖已经消失，这反映了黑洞的稳态性。

Now, inasmuch as comparison with the Bekenstein-Hawking entropy is meaningful only for macroscopic black holes, it is natural to assume that $a \gg 1$, and under this condition, $I(a)$ can be shown to have the following asymptotic behavior [17]:

现在，由于只有宏观黑洞才能和贝肯斯坦-霍金熵做有意义的比较，自然可以假设 $a \gg 1$，在此条件下，可以证明 $I(a)$ 满足如下渐近行为 [17]:

$$I(a) = \frac{\pi^2}{12}a + \mathcal{O}\left(\frac{1}{a}\right).$$

On the other hand, it is not difficult to see that

另一方面，不难证明

$$J(a) = \frac{1}{2a} + \mathcal{O}\left(\frac{1}{a^3}\right).$$

Putting everything together, we end up with

把所有内容结合起来，我们最终得到

$$< \mathbf{H}_{\text{link}} >= \frac{\pi^2}{6} + \mathcal{O}\left(\frac{1}{a^2}\right). \tag{16}$$

As the intersection of $\sum$ and $\mathcal{H}$ in two dimensions is just a point, the area law, if finite, should naturally turn out to be a pure number; therefore, (16), or the expected number of horizon molecules, is proportional to the area of the horizon in $1+1$.

由于二维中 $\sum$ 和 $\mathcal{H}$ 的交只是一个点，因此面积定律若有限，自然会是一个纯数；因此式 (16)，即视界分子的预期数量，正比于 $1+1$ 中视界的面积。

Some remarks about the above derivation of the area law in 2-d using this horizon molecule proposal are in order.

在这里，我们对使用该视界分子方案推导二维面积定律的过程做出一些说明。

The first remark concerns the locations of the pairs forming the molecules that give the dominant contribution to $< \mathbf{H}_{\text{link}} >$. It is easy to see that the dominant contribution to the integral $J(a)$ plainly comes from $r_{qq} \approx a$, but since r_{qq} is the radial coordinate r of sprinkled point q and since $r = a$ is the horizon, this implies that q resides near the horizon. Similarly, an inspection of the integral $I(a)$ shows that the dominant contribution to the integral $I(a)$ comes as well from $z \approx y \approx a$, which, since $z = r_{pp}$ and $y = r_{p0}$, implies in turn that sprinkled point p resides near the horizon as well [17]. Consequently, this counting can be said to be controlled by the near horizon geometry.

第一点说明涉及对 $< \mathbf{H}_{\text{link}} >$ 提供主要贡献的成分子对的位置。不难看出，对积分 $J(a)$ 的主要贡献显然来自 $r_{qq} \approx a$，但由于 r_{qq} 是撒点 q 的径向坐标 r，且 $r = a$ 就是视界，这意味着 q 位于视界附近。同理，对积分 $I(a)$ 的检查表明，积分 $I(a)$ 的主要贡献同样来自 $z \approx y \approx a$，结合 $z = r_{pp}$ 和 $y = r_{p0}$ 可得，撒点 p 也位于视界附近 [17]。因此可以说，该计数由近视界几何主导。

It should be noted too that from the unboundedness of the region $I^+(\sum) \cap I^+(\mathcal{H})$ and the finiteness of $< \mathbf{H}_{\text{link}} >$, we can infer that points q sitting arbitrarily close to the horizon but far from the $\sum \cap \mathcal{H}$ cannot continue to contribute indefinitely to $< \mathbf{H}_{\text{link}} >$. Moreover, the fact that $< \mathbf{H}_{\text{link}} >$ turns out to be just a pure number strongly suggests that the pairs which give the dominant contribution are not only residing near the horizon but are hovering near $\sum \cap \mathcal{H}$ too.

还应当注意，从区域 $I^{+}(\sum) \cap I^{+}(\mathcal{H})$ 的无界性和 $<\mathbf{H}_{\text{link}}>$ 的有限性，我们可以推出，任意靠近视界但远离 $\sum \cap \mathcal{H}$ 的点 q 无法持续对 $<\mathbf{H}_{\text{link}}>$ 做出贡献。此外，$<\mathbf{H}_{\text{link}}>$ 最终为纯数这一事实有力表明，提供主要贡献的粒子对不仅位于视界附近，还聚集在 $\sum \cap \mathcal{H}$ 附近。

It is interesting to look at this result and its features from another point of view. If we inspect the integral $I(a)$, we note that what makes the near horizon molecules special is the vanishing of the denominators in $I(a)$ when the dummy integration variables x and y tend to a. To the extent that it is this divergence which makes the horizon such a strong source for the links, here we may be reminded of the analogous fact that the strong redshift in the vicinity of the horizon allows modes of arbitrarily high (local) frequency to contribute to the entanglement entropy without influencing the energy as seen from infinity. Notice also that the clustering of p and q near the horizon is not simply a consequence of the maximality and minimality conditions we imposed on them. For instance, pairs (p, q) sitting arbitrarily close to the hypersurface $\sum$, with q arbitrarily close to the horizon, still do not contribute to the leading term in $I(a)$ if q is far from the horizon, namely, with coordinate $|u_p| \gg 1$.

从另一个角度审视这个结果及其特征是很有意义的。如果我们考察积分 $I(a)$，会发现近视界分子的特殊之处在于，当哑积分变量 x 和 y 趋近于 a 时，$I(a)$ 中的分母趋近于零。正是这种发散让视界成为连接的强来源，这一点让我们联想到一个类似的结论: 视界附近的强红移允许任意高 (局域) 频率的模式对纠缠熵做出贡献，而不会影响无穷远观测者测得的能量。还需要注意，p 和 q 聚集在视界附近并非我们对其施加的最大、最小条件的简单结果。例如，即使对对 (p, q) 任意靠近超曲面 $\sum$，且 q 任意靠近视界，只要 q 远离 a，即坐标 $|u_p| \gg 1$ 满足条件，它仍然不会对 $I(a)$ 的领头项做出贡献。

A Black Hole Far from Equilibrium: 2-Reduced Collapsing Null Matter

远离平衡态的黑洞: 二维约化坍缩零质

We now turn to another case which, though still spherically symmetric, is very far from equilibrium, namely, that of a spherically collapsing null shell of matter with stress energy tensor given by

我们现在转向另一个仍为球对称但远偏离平衡态的情形: 即球对称坍缩零物质壳，其能动张量由下式给出

$$T_{vv} = \frac{M\delta(b-v)}{4\pi r^2}$$

and the other components are identically zero.

其余分量均为零。

The collapsing shell forms a Schwarzschild BH. The Penrose diagram for the resulting spacetime (after dimensional reduction $S^2 \to$ point) is shown in Fig. 3. Let the shell sweep out the world sheet $v = b$ and let us choose for our hypersurface

坍缩壳形成史瓦西黑洞。维度约化 $S^2 \to$ 点后得到的时空彭罗斯图如图 3 所示。设该壳扫过世界面 $v = b$，我们为超曲面选取

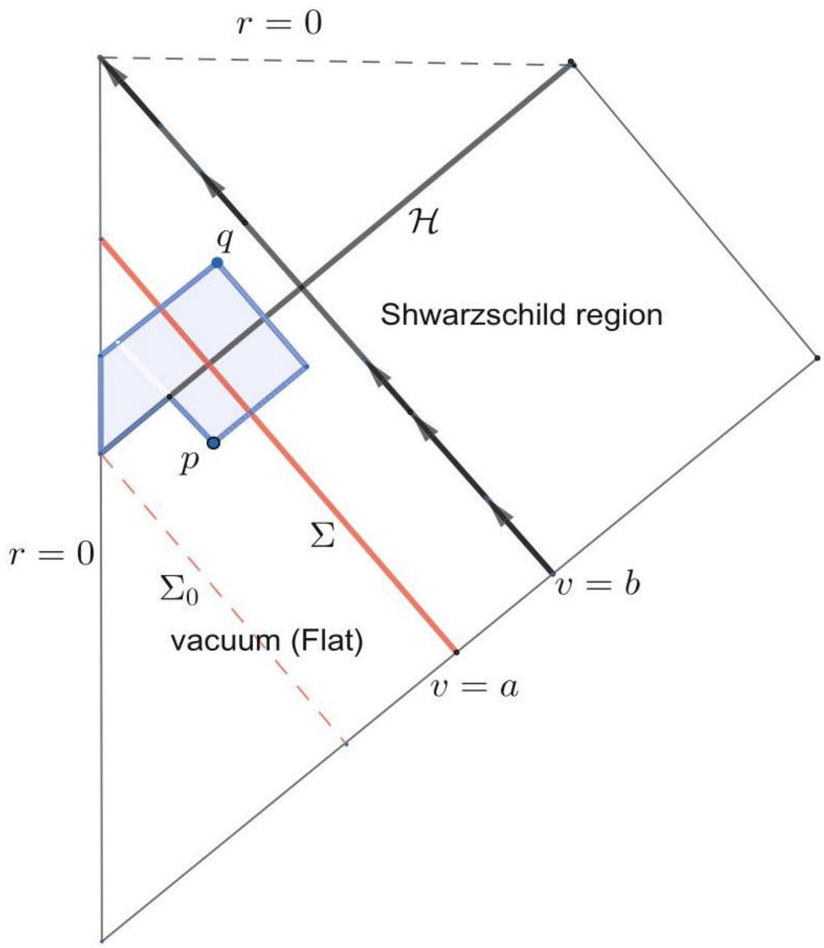


Fig. 3 A nonstationary BH. The region to the past of the world line of the infalling matter is a flat space with an expanding event horizon, whereas the one to its future is the Schwarzschild region. We have depicted an extra null hypersurface $\sum_0$ for later reference

图 3 非稳态黑洞。下落物质世界线过去一侧的区域是带有膨胀事件视界的平直空间，未来一侧则是史瓦西区域。我们画出了额外的零超曲面 $\sum_0$ 供后用

$\sum$ a second ingoing null surface defined by $v = a$, with $a < b$ so that $\sum$ lies wholly in the flat region. Here a is of course generally different from a defined in the Schwarzschild case, u and v are null coordinates, chosen so that the horizon first forms at $u = v = 0$ and normalized for convenience such that the line element in the flat region is given by

$\sum$ 第二个入射零曲面由 $v = a$ 定义，满足 $a < b$，因此 $\sum$ 完全位于平直区域。此处的 a 当然和史瓦西情形中定义的 a 不同，u 和 v 是零坐标，我们选取坐标使得视界首先在 $u = v = 0$ 处形成，并为了方便归一化，令平直区域的线元由下式给出

$$ds^2 = -2dudv + r^2 d\Omega^2.$$

Since our interest is again in macroscopic black holes, we will assume as before that the horizon radius at $\sum \cap \mathcal{H}$ is large in units such that $\varrho_c = 1$, which amounts to $a \gg 1$, and to simplify matters further, we will also restrict ourselves to a time well before the infalling matter arrives (as judged in the center of mass frame). One thus has the double inequality, $b \gg a \gg 1$. Once again, the calculation will be performed for the two-dimensional radial section rather than the full four-dimensional spacetime.

由于我们同样关注宏观黑洞，和之前一样我们假设 $\sum \cap \mathcal{H}$ 处的视界半径以 $\varrho_c = 1$ 为单位是大值，这等价于 $a \gg 1$； 为进一步简化，我们也将讨论范围限制在质心系中下落物质抵达之前很久的时刻。因此得到双重不等式 $b \gg a \gg 1$ 。我们同样将计算放在二维径向截面上进行，而非完整的四维时空。

Since we are assuming that the infalling matter is far to the future of the hypersurface $\sum$, points q sprinkled into that region should not contribute significantly when our minimality and link conditions are taken into account. For this reason, we shall, for convenience, restrict the counting to pairs (p, q) with $v_q < b$.

由于我们假设下落物质远在超曲面 $\sum$ 的未来，在考虑极小性条件和链接条件后，撒入该区域的点 q 贡献应当很小。因此为了方便，我们将计数限制在满足 $v_q < b$ 的对 (p, q) 上。

Using the definition of the horizon molecules we introduced above, one obtains for the expected number of horizon molecules

利用我们上文引入的视界分子定义，可以得到视界分子的期望数量为

$$< \mathbf{H}_{\text{links}} >= \int_a^b dv_q \int_0^{v_q} du_q \int_{-\infty}^0 du_p \int_0^a dv_p e^{-V} \tag{17}$$

where $V = u_q v_q - u_p \left(v_q - v_p\right) - u_q^2/2$, the volume of shaded region in Fig. 3.

其中 $V = u_q v_q - u_p \left(v_q - v_p\right) - u_q^2/2$ 为图 3 中阴影区域的体积。

Note here that the contribution of the points p with $v_p < 0$, i.e., to the past of $\sum_0$, has been ignored; we will return to its justification below.

此处注意我们忽略了满足 $v_p < 0$ 即位于 $\sum_0$ 过去的点 p 的贡献；我们会在下文说明该处理的合理性。

The integration over v_p and u_p is easy to perform, followed by change of variables, $x = v_q, y = v_q - u_q$, and we end up with

对 v_p 和 u_p 的积分很容易完成，随后进行变量替换 $x = v_q, y = v_q - u_q$ ，最终得到

$$< \mathbf{H}_{\text{link}} >= \int_a^b \ln\left(\frac{x}{x-a}\right) e^{-x^2/2} dx \int_0^x e^{y^2/2} dy. \tag{18}$$

At this stage it is not difficult to show that the leading behavior of this integral for large a is given by

在此阶段不难证明，当 a 很大时，该积分的领头阶行为为

$$< \mathbf{H}_{\text{link}} >= \frac{\pi^2}{6} - l\left(\frac{a}{b}\right) + \mathcal{O}\left(1/a^2\right) \tag{19}$$

where $l(x) \equiv \sum_{k=1}^{\infty} x^k/k^2$, a convergent series that vanishes in the limit $x \to 0$.

其中 $l(x) \equiv \sum_{k=1}^{\infty} x^k/k^2$ 是在极限 $x \to 0$ 下趋于零的收敛级数。

Originally the correction to the leading term in (19) were set to be of the order of $1/a$ in [17] and [18], but a careful repetition of the calculation due to Marr showed that the correction is of the order $1/a^2$ [19].

最初文献 [17] 和 [18] 认为 (19) 式领头项的修正量级为 $1/a$ ，但 Marr 重新仔细计算后发现修正量级为 $1/a^2$ [19]。

Since we have assumed that $a \ll b$, we can write this more simply as

由于我们已经假设 $a \ll b$ ，可以将其简化写为

$$< \mathbf{H}_{\text{link}} >= \frac{\pi^2}{6} + \mathcal{O}(a/b) + \mathcal{O}\left(1/a^2\right). \tag{20}$$

Notice that the presence of a negative contribution like $-l(a/b)$ was to be expected, since we have omitted to count molecules that extend past the shell into the Schwarzschild region. For $\sum$ near the shell, one obviously should not neglect such links, and this counting is incomplete. However, if the collapse is pushed far away from $\sum$, in particular to future infinity, we can safely restrict the counting to the flat region without worrying about the presence of the Schawrzschild region and therefore reducing the problem (even in higher dimension) to a counting in flat background geometry.

注意，像 $-l(a/b)$ 这样的负贡献本就在意料之中，因为我们没有统计延伸穿过壳层进入史瓦西区域的分子。对于靠近壳层的 $\sum$ ，显然不能忽略这类连接，因此目前的统计是不完整的。但如果坍缩发生在远离 $\sum$ 的位置，尤其是推进到未来无穷远，我们就可以放心地将统计范围限制在平直区域，无需考虑史瓦西区域的存在，从而将问题 (哪怕是高维情形) 简化为平直背景几何下的统计问题。

Now, what is striking about the above result is the occurrence of the same numerical coefficient $\pi^2/6$ in both (20) and (16). This agreement seems at first sight to furnish a nontrivial consistency check of the suggestion that one can attribute the horizon entropy to the horizon molecules made of ”causal links” crossing it.

现在，上述结果最引人注意的一点是，(20) 式和 (16) 式中都出现了相同的数值系数 $\pi^2/6$ 。乍看之下，这种一致性为以下观点提供了非平凡的一致性检验: 视界熵可以归因于由穿过视界的“因果连接”构成的视界分子。

As mentioned above, in writing (19) we implicitly ignored the contribution of pairs (p, q) with negative v_p. No justification for this was given in [17] nor in [18]. However, this point was raised and briefly discussed by Marr in [19].

如上所述，在写出 (19) 式时，我们默认忽略了 (p, q) 对中 v_p 为负的贡献。在文献 [17] 和 [18] 中都没有给出这一处理的正当性依据，但 Marr 在文献 [19] 中提出并简要讨论了这一问题。

It is easy to write an integral formula for this type of contribution, and maybe compute it; however, it is not difficult to argue that it should not be considered as part of the horizon molecules associated with $\sum \cap \mathcal{H}$. This kind of contribution counts the expected number of horizon molecules associated with a hypersurface $\sum_0$, Fig. 3, which is not intersecting the horizon, or they occur before the horizon formation. Therefore, they must be taken as sort of random statistical fluctuations extraneous and not genuine horizon molecules associated to $\sum$. Actually, if we remember that the geometrical setting we are using is 2-d reduced of a 4-dimensional one, this extra contribution would turn out to be just of order one in genuine 4-dimensional counting, thus a negligible fluctuation around the mean value.

我们很容易写出这类贡献的积分公式，或许也能计算它；但不难论证，这类贡献不应当被算作与 $\sum \cap \mathcal{H}$ 关联的视界分子的一部分。这类贡献统计的是与超曲面 $\sum_0$ (见图 3) 关联的视界分子的预期数量，这类超曲面不与视界相交，或是这类贡献出现在视界形成之前。因此，它们必须被视作无关的一类随机统计涨落，并非与 $\sum$ 关联的真正视界分子。实际上，如果我们记得我们所用的几何框架是四维时空的二维约化版本，那么这个额外贡献在真正的四维统计中量级仅为 1，因此是均值周围可以忽略的涨落。

On the Min/Max Conditions

论最大/最小条件

As we briefly discussed before picking up the particular choice for the "max/min" conditions we adopted in the definition the causal link proposal, this choice did not seem unique or particularly sacred and other variants were possible. Of course, one must be careful not to use something like " q minimal in $I^+(\sum)$," which would drive $< \mathbf{H}_{\text{link}} >$ to zero in the limit of null $\sum$, but this does not rule out, for example, a condition like " p maximal in $I^-(\sum)$."

正如我们在确定因果联络方案定义中所采用的“最大/最小”具体条件之前简要讨论过的，这一选择并非唯一也不是绝对不可变动，其他变体也是可行的。当然，必须注意不要采用诸如“在 $I^+(\sum)$ 中 q 最小”这样的表述，这会在零 $\sum$ 极限下将 $< \mathbf{H}_{\text{link}} >$ 驱至零，但这并不排除例如“在 $I^-(\sum)$ 中 p 最大”这类条件。

Let us note that it turns out that there are at least two variants of the "max/min" condition that seem to be equivalent, as long as the leading term is concerned. These variants are

我们发现，至少存在两种等价的“最大/最小”条件变体，就领头项而言二者是等价的。这些变体分别是

Variant 1: p is max in $I^-(\sum)$ and q is min in $I^+(\mathcal{H}) \cap I^+(\sum)$.

变体 1: p 在 $I^-(\sum)$ 中取最大值，且 q 在 $I^+(\mathcal{H}) \cap I^+(\sum)$ 中取最小值。

Variant 2: p is max in $I^-(\sum) \cap I^-(\sum)$ and q is min in $I^+(\mathcal{H}) \cap I^+(\sum)$.

变体 2: p 在 $I^-(\sum) \cap I^-(\sum)$ 中取最大值，且 q 在 $I^+(\mathcal{H}) \cap I^+(\sum)$ 中取最小值。

If we consider, for instance, the first variant, it is easy to show that the resulting expected number of horizon molecules has the same asymptotic behavior as the one resulting from the original causal link proposal [17], namely,

例如，若我们考虑第一种变体，不难证明其得到的视界分子期望数与原始因果联络方案 [17] 得到的结果具有相同的渐近行为，即

$$< \mathbf{H}_{\text{link}} >= \frac{\pi^2}{6} + O\left(1/a^2\right).$$

Thus, for this variant at least, one obtains exactly the same numerical answer as the original proposal we started with. As for the second variant, although it has not been worked out explicitly, we do not expect the slight change in the volume $V(p,q)$ should alter the leading term.

因此，至少对该变体而言，所得的数值结果与我们最初的原始方案完全一致。至于第二种变体，尽管尚未经过明确推导，我们认为体积 $V(p,q)$ 的微小变动不会改变领头项。

Another related feature the link counting must have if it is to yield the horizon area is that, within reason, the expected number of horizon molecules should depend only on the intersection $\mathcal{H} \cap \sum$ and not on how the surface $\sum$ is prolonged outside or (especially) inside the horizon $\mathcal{H}$. For example, one should get the same answer for both of the continuations shown in Fig. 4. The case where the difference is confined to the interior black hole region is of particular significance for the entanglement interpretation of horizon entropy, since such a difference cannot, by definition, influence the effective density operator for the external portion of $\sum$ (at least to the extent that unitary quantum field theory is a good guide). For instance, we note that the volume $V(p,q)$ needed to insure p maximal in $I^-(\mathcal{H}) \cap I^-(\sum)$ and q be minimal in $I^+(\mathcal{H})$ is the same for both $\sum_{\text{ext}} \cup \sigma_1$ and $\sum_{\text{ext}} \cup \sigma_2$; therefore, from this perspective the definition we have so far adopted seems to have advantage over the other two variants, at least in the case of null $\sum$. However, in view of the fact that the leading order is controlled by contributions coming from links residing near the horizon, we expect the different variants to have the same leading behaviors no matter how $\sum$ is prolonged inside or outside the horizon. Indeed, the issue of which max/min condition is favored cannot be settled nor properly discussed unless we settle the central issue of how to define the horizon molecules in a way that works in higher dimensions and for both types of hypersurfaces, null and spacelike.

若要让联络计数得到视界面积，它还需要满足另一个相关性质: 合理范围内，视界分子的期望数应当仅依赖于交截 $\mathcal{H} \cap \sum$，与曲面 $\sum$ 如何向外延拓、尤其是如何向视界 $\mathcal{H}$ 内部延拓无关。例如，对图 4 所示的两种延拓方式，应当得到相同结果。差异仅局限于黑洞内部区域的情况对视界熵的纠缠诠释尤为重要，因为根据定义，这种差异不会影响 $\sum$ 外部部分的有效密度算符 (至少在幺正量子场论适用的范围内是如此)。例如，我们注意到，为保证“在 $I^-(\mathcal{H}) \cap I^-(\sum)$ 中 p 最大且在 $I^+(\mathcal{H})$ 中 q 最小”所需的体积 $V(p,q)$ 对 $\sum_{\text{ext}} \cup \sigma_1$ 和 $\sum_{\text{ext}} \cup \sigma_2$ 而言是相同的；因此从这个角度看，我们目前采用的定义比另外两种变体更有优势，至少在零 $\sum$ 的情况下是这样。不过，鉴于领头阶由来自视界附近的联络贡献主导，我们预期无论 $\sum$ 向视界内外如何延拓，不同变体的领头行为都是一致的。事实上，除非我们解决核心问题: 找到一种能同时适用于高维、适用于类光和类空两类超曲面的视界分子定义方式，否则我们无法解决也无法恰当讨论哪一种最大/最小条件更优的问题。

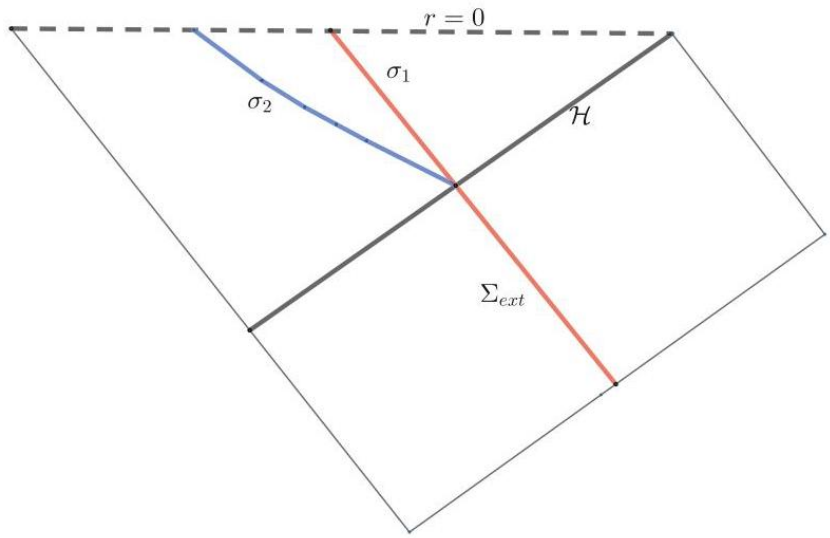


Fig. 4 Two continuations of a hypersurface to the interior region

图 4 一张超曲面向内部区域的两种延拓方式

The Spacelike Hypersurface in 2-D Reduced

二维约化中的类空超曲面

So far the counting of the horizon molecules has been restricted to null hypersur-faces in 2-d reduced black hole geometries. However, no proposal can be considered fully successful, even in two dimensions, unless it correctly reproduces the same result for both null and spacelike hypersurfaces.

到目前为止，视界分子计数一直仅针对二维约化黑洞几何中的类光超曲面。但哪怕是在二维中，任何方案都不能算是完全成功的，除非它能对类光和类空超曲面都给出一致的正确结果。

In this section, we look at this issue by discussing the previous link counting when $\sum$ is a spacelike hypersurface crossing the horizon under the same max/min conditions.

在本节中，我们探讨这一问题：当 $\sum$ 是满足相同最大/最小条件、穿过视界的类空超曲面时，我们讨论之前的关联计数。

It is first intriguing to discuss one of the heuristic arguments that is sometimes invoked in this context to conclude that the null and spacelike counting should be expected to yield the same result.

首先值得探讨的是这一语境下常被引用的一个启发式论证，该论证认为类光和类空计数理应得到相同结果。

This argument generally goes as follows, [18,21].

该论证大致表述如下，参见文献 [18,21]。

Consider a one-parameter family of spacelike hypersurfaces $\sum_t$ which continuously deform to a null hypersurface, $\sum = \lim_{t\to\infty} \sum_t$. On the other hand, the region one sprinkles into and so the probability measure are also continuous with respect to the deformations. Now, because all spacelike hypersurfaces give the same result, then the null hypersurface $\sum$ which can be casted as a limit of a sequence of spacelike hypersurface should also give the same result. In the flat case one can equally use Lorentz invariance of the counting and the fact any spacelike hypersurface must give the same result, as any other related to it by a boost, and in the limit of tilting, a spacelike line becomes null. Note that a similar argument can also be made in the Schwarzschild case, using time-translation Killing vector instead of the boost killing vector.

考虑单参数族类空超曲面 $\sum_t$，它们可以连续形变为类光超曲面 $\sum = \lim_{t\to\infty} \sum_t$。另一方面，我们随机撒入的区域以及概率测度也会随形变连续变化。由于所有类空超曲面都给出相同结果，因此可作为一系列类空超曲面极限得到的类光超曲面 $\sum$ 理应也给出相同结果。在平直情形中，我们同样可以利用计数的洛伦兹不变性，以及任意两个通过 boost 联系起来的类空超曲面必然给出相同结果这一事实；而在倾斜的极限下，类空线会变为类光线。请注意，在史瓦西情形中也可以给出类似论证，只需用平移时间的基林矢量代替 boost 基林矢量即可。

Stated mathematically, one would expect the following limit to hold:

用数学表述的话，我们预期下述极限成立：

$$\lim_{t\to\infty} < \mathbf{H}\left(\sum_t\right) >=< \mathbf{H}\left(\sum\right) > . \tag{21}$$

In the above equation we do not of course require strict equality, but it would be enough to hold in the limit $l_c \to 0$ modulo some statistical deviations from the leading mean value. In [21], it was, for instance, argued that it is the non-commutativity of the two limits, $t \to \infty$ and $l_c \to 0$, which causes the above identity to fail, namely,

当然，在上式中我们不要求严格相等，只要该式在 $l_c \to 0$ 的极限中成立就足够了，允许结果偏离主导均值存在一定统计偏差。例如，文献 [21] 中就指出，正是两个极限 $t \to \infty$ 和 $l_c \to 0$ 不对易，导致上述恒等式不成立，即：

$$\lim_{l_c\to 0}\lim_{t\to\infty} l_c^{d-2} < \mathbf{H}\left(\sum\nolimits_t\right) > \neq \lim_{t\to\infty}\lim_{l_c\to 0} l_c^{d-2} < \mathbf{H}\left(\sum\nolimits_t\right) > . \tag{22}$$

However, we shall see in the fourth section that in some horizon molecules counting the limit $l_c \to 0$ is not at all required for the derivation of the area law; nonetheless, the null and spacelike hypersurfaces give two different results. Therefore, we conclude that the above heuristic argument invoking the non-commutativity of the two limits is at best not generally sustainable, as some counting could be inherently discontinuous and depend on the nature of the hypersurface crossing the horizon.

但我们会在第四节看到，对某些视界分子计数而言，推导面积定律根本不需要极限 $l_c \to 0$；尽管如此，类光和类空超曲面仍会给出两个不同结果。因此我们总结认为，上述引入两个极限不对易的启发式论证往好了说也不具备普遍适用性，因为某些计数本身就是不连续的，依赖于穿过视界的超曲面本身的类型。

Let us now consider a spacelike $\sum$ given by $t = \frac{a}{2}$. To facilitate the discussion it is convenient to introduce a null hypersurface $\sum'$ defined by $v = a$. Again, we shall push the collapse to future infinity and restrict the counting to the flat region, Fig. 5.

现在我们考虑由 $t = \frac{a}{2}$ 给出的类空超曲面 $\sum$ 。为便于讨论，我们引入由 $v = a$ 定义的类光超曲面 $\sum'$ 。我们再次将坍缩推到未来无穷远，并将计数限制在平直区域，参见图 5。

It is not difficult to see that one has to distinguish five cases, each having a different expression for the volume $V(p, q)$. These cases are depicted in Fig. 5.

不难看出我们需要区分五种情形，每种情形对应体积 $V(p, q)$ 的不同表达式。这些情形已呈现在图 5 中。

The contributions $A1, A2, A3$ can be easily seen to be qualitatively of the same order of magnitude as the null contribution we already evaluated; thus, they will just give constants of order one, but surely each is less than $\frac{\pi^2}{6}$.

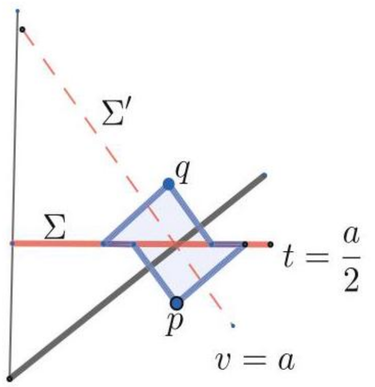
容易看出，贡献 $A1, A2, A3$ 在量级上与我们已经计算过的类光贡献定性一致；因此它们只会给出一阶常数，而且每个贡献都必然小于 $\frac{\pi^2}{6}$ 。

Contributions of type (B) and (C) are different and one cannot directly conclude that they are finite or of the order one. For instance, contributions from pairs (p, q) with $u_q \to 0$ and $t_p \to a/2$ could lead to IR divergences. However, it was explicitly shown in [17] that both contributions are finite and of order one.

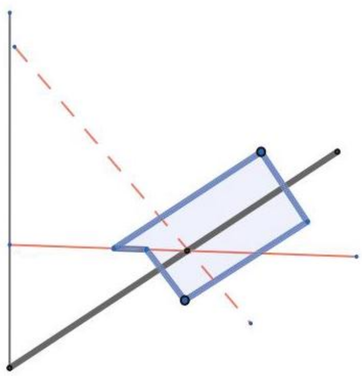
(B) 型和 (C) 型的贡献并不相同，我们无法直接得出它们是有限或一阶的结论。例如，满足 $u_q \to 0$ 和 $t_p \to a/2$ 的对 (p, q) 的贡献可能会导致红外发散。但文献 [17] 已经明确证明，这两种贡献都是有限的，且量级为一阶。

Now, although it was possible to show that the expected number of such horizon molecules gives a constant of order one, the question whether the spacelike and null cases yield the same result has so far remained open due to the analytical intractability of the integrals involved. But what should be noted in this context is that the difficulty to settle this issue in two dimensions may not solely be of technical nature, due to the intractability of the integrals, but there could be another issue of conceptual nature at work here. For example, in the nonequilibrium case and for a null hypersurface counting, we argued that contributions coming from the pairs (p, q) with $v_p < 0$ are to be viewed as random fluctuations around the mean value not genuinely associated with $\sum \cap \mathcal{H}$ not to be included in the counting, despite the fact that they are not zero, but in higher dimension they are easily seen to be negligible for a macroscopic horizon. For the same reasons we expect similar fluctuations to be present in the above different contributions; in particular we do not expect arrangement of type- (B) and (C) to be fully associated with $\sum \cap \mathcal{H}$. However, as it will be shown in the next section, the present link counting fails to work beyond two dimensions due to IR divergences, and therefore, pursuing this issue would be no more than mathematical curiosity without physical guidance nor relevance.

目前，虽然可以证明这类视界分子的期望数为一阶常数，但由于涉及积分解析难解，类空情形与类零情形是否给出相同结果这一问题至今仍未解决。不过在此背景下需要注意，二维中该问题难以解决或许不只是积分难解带来的技术层面问题，这里可能还存在另一个概念层面的问题。例如，在非平衡情形与类零超曲面计数中，我们提出，来自 (p, q) 和 $v_p < 0$ 的贡献应被视为均值附近的随机涨落，并非真正与 $\sum \cap \mathcal{H}$ 相关，因此不应纳入计数——尽管这些贡献不为零，但在高维情形中它们对宏观视界而言显然可以忽略。同理，我们预计上述不同贡献中也存在类似涨落；特别地，我们不认为 (B) 型和 (C) 型排布完全关联于 $\sum \cap \mathcal{H}$ 。但如下一节所示，由于红外发散，现有链路计数在二维以上无法成立，因此深究该问题不过是缺乏物理指引与物理相关性的数学猎奇罢了。



Arrangement type-A1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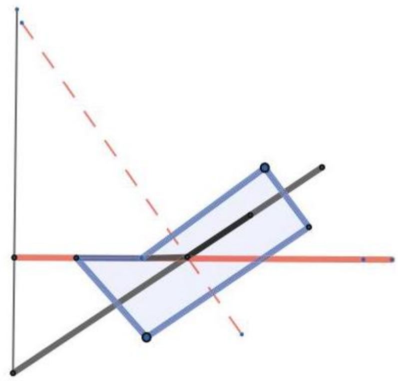
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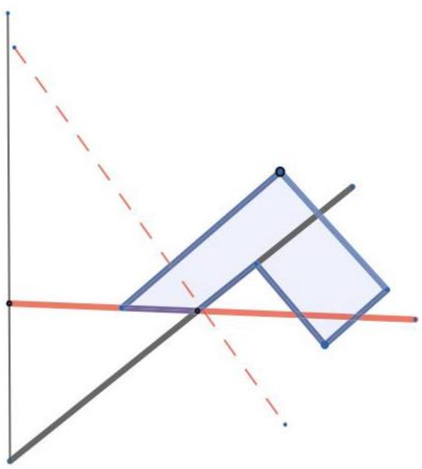

Arrangement type-B

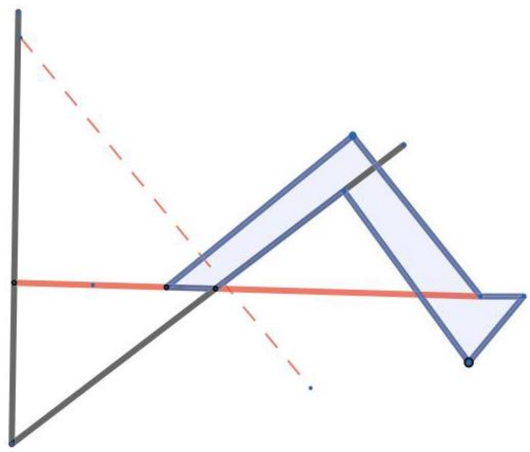

Arrangement type-C

Fig. 5 Different arrangement contributing in the spacelike case, according to different volume expressions. For the arrangement type-A1 the point q does not necessary lie to the future of $\sum'$; it could be to the past of it as well

图 5 类空情形中，不同排布根据不同体积表达式产生贡献。对于 A1 型排布，点 q 不一定位于 $\sum'$ 的未来，它也可以位于 $\sum'$ 的过去

The Failure of the Causal Link Proposal in Higher Dimensions

因果连接方案在高维下的失效

In view of the success and promising results of the causal link counting in two-dimensional models, the natural step would of course be to try to apply it on a more realistic four-dimensional black hole background. Any direct attempt to do this counting in Schwarzschild geometry will inevitably be encountered by mathematical complications that are almost impossible to surpass. However, the results obtained previously in two dimensions enable us to transform the whole problem into calculation in 4-dimensional (or d-dimensional) flat spacetimes using the collapsing null-shell model by pushing the collapse world-line to future infinity. Therefore, one is in principle entitled to consider a flat spacetime with the future light cone of the origin being the horizon, take a hypersurface $\sum$ (spacelike or null) intersecting the horizon, and compute the expected number of horizon molecules, Fig. 6.

鉴于二维模型中因果连接计数取得了成功且前景可观，自然下一步当然是尝试将其应用到更贴近实际的四维黑洞背景中。在史瓦西几何中直接进行这类计数，无可避免会遇到几乎无法克服的数学难题。不过，二维中先前得到的结果让我们可以通过坍缩零壳模型，将整个问题转化为四维 (或 d 维) 平直时空的计算——只要把坍缩世界线推至未来无穷远。因此，原则上我们可以考虑这样一个平直时空: 原点的未来光锥就是视界，取一个与视界相交的超曲面 $\sum$ (类空或类光)，计算视界分子的预期数量，见图 6。

Although working in flat spacetimes drastically simplifies the problem, in $d > 2$ the calculation of $< \mathbf{H}_{\text{link}} >$ is still complicated enough and a much more elaborated technique is needed to do the counting explicitly, for both the null and spacelike hypersurfaces. The calculation of the volumes needed to insure the link and max/min conditions is lengthy, and it turns out that one has to distinguish many cases depending on the relative positions of the points p and q , each case making its own contribution to $< \mathbf{H}_{\text{link}} >$ [17]. For instance, for spacelike $\sum$ and to the exception of one contribution, coming from an arrangement similar to type A. 1 in two dimensions, Fig. 5, which could be evaluated and was reported in [17, 18], the remaining arrangements turned out to be either very complicated or intractable. Nonetheless, at least for one nontrivial arrangement the volume was computed exactly in [17]. The arrangement in question is of the type-B, depicted in Fig. 5 (its four-dimensional analogue). For this particular arrangement it was later realized by the author that its corresponding contribution to $< \mathbf{H}_{\text{link}} >$ diverges. It is unnecessary to give the detail of this calculation, but it is not difficult to qualitatively understand the source of this IR divergence.

尽管在平直时空工作大幅简化了问题，但在 $d > 2$ 中对 $< \mathbf{H}_{\text{link}} >$ 的计算仍然相当复杂，无论对类光还是类空超曲面，都需要复杂得多的技术才能完成显式计数。为满足连接条件和最大/最小条件所需的体积计算十分冗长，结果表明我们必须根据点 p 和 q 的相对位置区分多种情形，每种情形都会对 $< \mathbf{H}_{\text{link}} >$ 做出贡献 [17]。例如，对于类空 $\sum$，除了一种贡献外——该贡献来自一种类似二维 A 型 1 排布的构型，见图 5，其结果可计算并已记载于 [17,18]——其余排布要么极其复杂要么难以处理。尽管如此，文献 [17] 中至少对一种非平凡排布精确计算出了体积。我们讨论的该排布属于 B 型，描绘在图 5 中 (它的四维类比)。作者后来发现，这一特定排布对 $< \mathbf{H}_{\text{link}} >$ 的对应贡献是发散的。我们无需给出该计算的细节，但不难定性理解这种红外发散的来源。

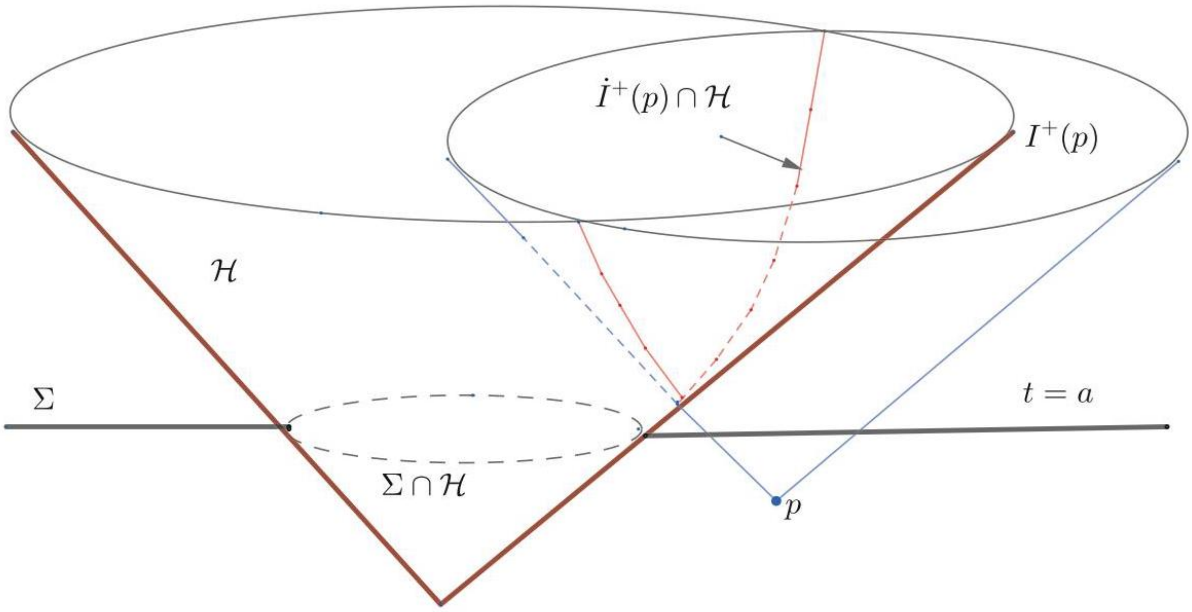


Fig. 6 The red curve shows the intersection of $\dot{I}^+(p)$ with the horizon; this curve extends to future infinity and by considering sprinkled points q asymptotically approaching this curve, an arbitrary number of $p \prec \cdot q$ links can be found

图 6 红色曲线显示了 $\dot{I}^+(p)$ 与视界的交线；该曲线延伸至未来无穷远，通过考虑渐近逼近该曲线的洒放点 q，可以找到任意数量的 $p \prec \cdot q$ 连接

Let us first take a step back and consider the arrangement type-B depicted in Fig. 5, in its $1+1$ version. The only potential source of IR divergences comes from the contributions of points q arbitrarily far away from the intersection point $\sum \cap \mathcal{H}$; however, the e^{-V} term appearing in the integrand exponentially suppresses all contributions except those when p is arbitrarily close to the intersection point and q bound to the horizon, which is not enough to produce any IR divergences.

我们先退一步，考虑图 5 中描绘的 B 型排布的 1+1 维版本。红外发散的唯一潜在来源是距离交点 $\sum \cap \mathcal{H}$ 任意远的点 q 的贡献；不过，被积函数中出现的 e^{-V} 项会指数抑制所有贡献，只有当 p 任意靠近交点且 q 束缚在视界上时除外，这不足以产生任何红外发散。

The situation in higher dimensions is quite different and this can be grasped by considering the 2+1 case depicted in Fig. 5 and using the qualitative argument given in [19].

高维下的情况完全不同，这一点可以通过考虑图 5 描绘的 2+1 维情形，结合文献 [19] 给出的定性论证理解。

In Fig. 6 the horizon light cone $\mathcal{H}$ is intersected by a spacelike hypersurface $\sum$, $t = a$ say. Consider a point p and let $\dot{I}^+(p)$ be the boundary of its future light cone. Unlike $1+1$, in $2+1$ dimensions (or higher) the intersection of $\mathcal{H}$ is no longer a point, but rather a curve ($d-2$ -dimensional surface in general). This adds a new degree of freedom and allows the existence of new links formed with points q , asymptotically close to $I^+(p) \cap \mathcal{H}$ and arbitrarily far form $\sum \cap \mathcal{H}$. In addition, and unlike the $1+1$ case, the point p is not at all required to be arbitrarily close to the $\sum \cap \mathcal{H}$ for the volume V to vanish; it is enough to be arbitrarily close to $\sum$ and anywhere far from the intersection of $\mathcal{H}$ and $\sum$. For these distant pairs (p, q) , the interval between p and q remains small and highly likely to be free from additional sprinkled points. For these reasons the e^{-V} is not enough to suppress the contributions of such pairs, as there is an infinite number of potential pairs (p, q) with vanishing volume in the analytic limit. As a consequence the expected number of causal links, no matter which max/min conditions are imposed, will diverge like Λ_r^{d-2} , in d -dimensions, with Λ_r an appropriate IR cutoff. This IR divergence is incurable within the causal link proposal and a real departure from the link definition is therefore inevitable.

在图 6 中，视界光锥 $\mathcal{H}$ 被类空超曲面 $\sum$ (例如 $t = a$) 所截。取一点 p ，令 $\dot{I}^+(p)$ 为其未来光锥的边界。与 $1+1$ 不同，在 $2+1$ 维 (及更高维) 空间中，$\mathcal{H}$ 的截线不再是一个点，而是一条曲线 (一般为 $d-2$ 维曲面)。这引入了一个新的自由度，允许与渐近距离 $I^+(p) \cap \mathcal{H}$ 、且任意远离 $\sum \cap \mathcal{H}$ 的点 q 形成新连接。此外，与 $1+1$ 的情形不同，要使体积 V 消失，点 p 完全不需要任意靠近 $\sum \cap \mathcal{H}$ ；它只需任意靠近 $\sum$ ，且位于远离 $\mathcal{H}$ 与 $\sum$ 交点的任意位置即可。对于这些远距离对 (p, q) ， p 与 q 之间的间隔仍然很小，极大概率不会包含额外的撒入点。因此，e^{-V} 不足以抑制这类对的贡献，因为在解析极限下存在无穷多体积为零的潜在对 (p, q) 。结果就是，无论施加何种最大/最小条件，因果连接的预期数量在 d 维空间中都会按 Λ_r^{d-2} 发散，其中 Λ_r 是合适的红外截断。这种红外发散在因果连接方案的框架内无法解决，因此偏离原有连接定义是不可避免的。

The Triplet Proposal

三元组提案

The failure of the causal link proposal beyond $1+1$ subsequently led soon to different and modified causet structures as new candidates for the horizon molecules.

因果关联提案在 $1+1$ 之外失效，随后很快催生了不同的改进型因果集结构，成为视界分子的新候选。

We first note that it is almost obvious that the day cannot be saved by simple modifications of the link proposal, for instance, by modifying the max/min conditions, as we have exhausted all acceptable variants. The first attempt to depart from the link structure was taken by Marr [19]. This attempt was mainly based on the ”triad” structure or ”triplet.” Although there are some hints that this modified horizon molecular structure may fail to cure all the IR divergences we observed in the causal link proposal in higher dimensions, we find that the triplet proposal is worthy of brief discussion. We shall omit all technical details, as it is similar in spirit to the link counting and only focus on the main results and their discussion.

我们首先指出，显然仅对关联提案做简单修改 (例如调整最大/最小条件) 无法解决问题，因为我们已经穷尽了所有可接受的变体。马尔 [19] 首次尝试脱离关联结构，这项尝试主要基于“三元组”结构，也就是是“三元组提案”。虽然已有迹象表明，这种改进后的视界分子结构未必能解决我们在高维因果关联提案中观测到的所有红外发散问题，但我们认为三元组提案仍值得简要讨论。由于其核心思路与关联计数类似，我们将省略所有技术细节，仅聚焦主要结果及其讨论。

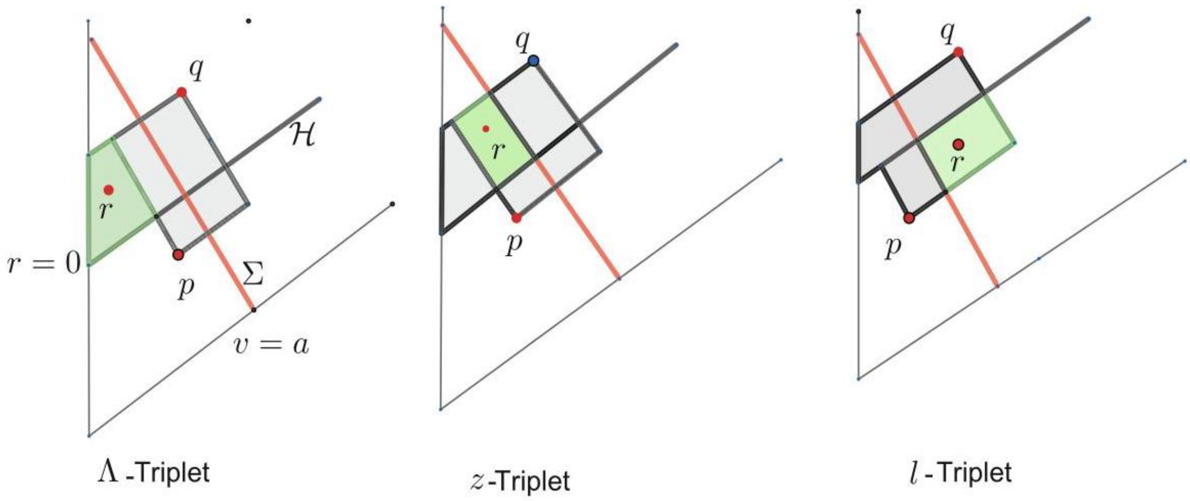


Fig. 7 The regions shaded grey are required to be free from any sprinkled points, while the green ones have only one point

图 7 灰色阴影区域要求不存在任何撒播点，绿色区域仅存在一个点

Let us start by noting that there actually some suggestion that a certain type of triplets is naturally related to the kind of correlation responsible for entanglement entropy in a quantum field theory framework [26].

首先我们注意到，已有观点表明，特定类型的三元组自然关联于量子场论框架中引发纠缠熵的那类关联 [26]。

Λ -Triplet (Marr 2007)

Λ -三重态 (Marr 2007)

A horizon molecule with respect to a given hypersurface $\sum$ is a triplet (p,q,r) satisfying the following conditions:

给定超曲面 $\sum$ 对应的视界分子是满足以下条件的三元组 (p,q,r)：

1. $p \in I^-(\sum) \cap I^-(\mathcal{H})$.
2. $q \in I^+(\sum) \cap I^+(\mathcal{H})$.
3. $r \in I^-(\sum) \cap I^+(\mathcal{H})$.

4. $|I[p,q]| = 0, |I[r,q]| = 0$.
5. p is maximal in $I^-(\sum) \cap I^-(\mathcal{H})$, q is minimal-but-one in $I^+(\mathcal{H})$, and r is minimal in $I^-(\sum) \cap I^+(\mathcal{H})$
.

5. p 在 $I^-(\sum) \cap I^-(\mathcal{H})$, q 中是极大元，在 $I^+(\mathcal{H})$ 中是次极小元，且 r 在 $I^-(\sum) \cap I^+(\mathcal{H})$ 中是极小元。

Notice that the 4th condition automatically requires p and r to be causally unrelated or spacelike related.

注意，第 4 个条件自动要求 p 和 r 无因果关联或类空关联。

The condition q minimal-but-one in $I^+(\mathcal{H})$ is meant to ensure that the only point in $I^-(q)$ is r , Fig. 7.

“q 在 $I^+(\mathcal{H})$ 中为次极小元”这一条件的作用是保证 $I^-(q)$ 中仅存在点 r ，见图 7。

Marr first applied the Λ -triplet proposal for the collapsing shell model in $1+1$ - pushing the collapsing shell to future infinity- to obtain an expected number of horizon molecules of order one; more precisely she obtained

Marr 首次将 Λ -三重态方案应用于 $1+1$ 中的坍缩壳模型——将坍缩壳推至未来类空无穷远——得到视界分子的期望数量为一阶；更准确地说，她得到

$$< \mathbf{H}_{\Lambda-\text{ triplet}} >= \frac{\pi^2}{6} - 1 + \mathcal{O}\left(\left(a/b\right)^2\right). \tag{23}$$

Using a triplet instead of a simple link (doublet) has therefore reduced the expected number of horizon molecules by one.

因此，使用三重态而非简单链接 (二重态) 将视界分子的期望数量减少了 1。

In contrast to the collapsing shell model in $1+1$, the case of 2-d reduced Schwarzschild BH turned out to be analytically intractable for the Λ -triplet.

与 $1+1$ 中的坍缩壳模型不同，二维约化史瓦西黑洞对于 Λ -三重态而言是解析不可解的。

As mentioned earlier, the underlying motivation that led to the departure from the link structure to the triplets was to kill off the contributions coming from $p-q$ links generated by points q arbitrarily close to $\mathcal{H} \cap \dot{I}(p)$ but arbitrarily far from $\mathcal{H} \cap \sum$. However, it has been argued in [19] that although the introduction of a third element in the horizon molecule structure, i.e., r , seems to cure this IR divergences, the Λ - triplet counting still suffers from another IR divergence, of course already present in the link proposal. This IR divergence arises as a result of having unsuppressed contributions coming now from points p asymptotically approaching $\sum \cap \dot{I}(q)$ as they move further into the past, simultaneously keeping the $p-q$ interval small, and as r does not bound p from $\sum$, they are spacelike related, thus avoiding any exponential suppression.

如前所述，我们从链接结构转向三元组结构的根本动机是消除由任意接近 $\mathcal{H} \cap \dot{I}(p)$ 但任意远离 $\mathcal{H} \cap \sum$ 的点 q 生成的 $p-q$ 链接带来的贡献。然而，文献 [19] 指出，尽管在视界分子结构中引入第三个元素即 r 似乎解决了这种红外发散，但 Λ -三元组计数仍然存在另一种红外发散，这种发散其实早已存在于链接方案中。该红外发散的产生原因是: 当点 p 向过去不断运动渐近趋近 $\sum \cap \dot{I}(q)$，同时保持 $p-q$ 间隔很小，此时这些点的贡献未受到抑制；并且由于 r 无法从 $\sum$ 方向限制 p，二者呈类空关联，因此避免了任何指数抑制。

The above qualitative argument seemingly rules out the Λ -triplet as a possible alternative candidate that would work in higher dimensions and led Marr to consider other possible arrangements for the triplet. The guide of course was to cure the IR divergences which plagued the link counting and persisted in the Λ -triplet counting.

上述定性论证似乎排除了 Λ -三重态作为适用于更高维的备选候选方案，这促使 Marr 考虑三重态的其他可能排列。当然，研究的核心目标是解决困扰链接计数、且在 Λ -三重态计数中依然存在的红外发散问题。

To that end two different arrangements were considered in [19], the z - and l - triplet.

为此，文献 [19] 考虑了两种不同排列，即 z -三重态和 l -三重态。

The z -triplet is obtained from the definition of the Λ -triplet by keeping the first three conditions, moving r to the future of p to form a link with it, keeping its link relation with q, so the three points form a path or a maximal chain, i.e., $p \prec \cdot r \prec \cdot q$, and removing the minimality condition on r from the 5th condition, Fig. 7.

z -三重态由 Λ -三重态的定义修改而来: 保留前三个条件，将 r 移至 p 的未来以和 p 形成链接，保留它与 q 的链接关系，因此三个点形成一条路径或一条极大链，即 $p \prec \cdot r \prec \cdot q$，并从第 5 个条件中移除对 r 的极小性条件，见图 7。

As for the l -triplet, it is a rearrangement of the z -triplet by moving r to region $\in I^+(\sum) \cap I^-(\mathcal{H})$, requiring p to be maximal in $I^-(\sum) \cap I^-(\mathcal{H})$ and q maximal in $I^+(\mathcal{H})$, Fig. 7.

而 l -三重态是 z -三重态的重排: 将 r 移至区域 $\in I^+(\sum) \cap I^-(\mathcal{H})$，要求 p 在 $I^-(\sum) \cap I^-(\mathcal{H})$ 中为极大元，且 q 在 $I^+(\mathcal{H})$ 中为极大元，见图 7。

Marr used both the l - and z -triplet to count the expected number of horizon molecules for the collapsing null shell $1+1$ reduced model and obtained the following results:

马尔使用 l 三重态和 z 三重态计算了坍缩零壳 $1+1$ 约化模型的视界分子预期数量，得到了如下结果:

$$< \mathbf{H}_{z-\text{ triplet}} >= 2 - \frac{\pi^2}{6} + \mathcal{O}\left((a/b)^2\right) \tag{24}$$

$$< \mathbf{H}_{\text{l-triplet}} >= 1 + \mathcal{O}(a/b). \tag{25}$$

Moreover, the l -triplet turned out to be manageable analytically for the $1+1$ Schwarzschild static model and gave the same result (leading term) as the collapsing null shell setting, which is a promising result.

此外，对于 $1+1$ 施瓦西静态模型，l 三重态可进行解析处理，且得到的领头项结果与坍缩零壳情形一致，这是一个很有前景的结论。

Let us remember that both the z - and l -triplet were introduced with an eye on their use as a candidate for horizon molecules in higher dimensions. Although Marr did not report any analytical results concerning the triplet structures in $1+2$ or $1+3$, she gave qualitative argument suggesting that the z - and l -triplets are in principle free of the IR divergences we discussed above. Her argument goes as follows: the presence of a third element r in the z -triplet bounds p away from $\sum$ and q from $\mathcal{H}$, whereas in the l -triplet the role of r is reversed; it bounds p away from $\mathcal{H}$ and q from $\sum$. Hence, for both triplets an arbitrarily large number of $p-q$ links are unlikely to build up in higher dimensions.

我们需要记得，z 三重态和 l 三重态的引入初衷是作为更高维下视界分子的候选结构。尽管马尔没有给出 $1+2$ 或 $1+3$ 中三重态结构的任何解析结果，但她给出了定性论证表明，原则上 z 三重态和 l 三重态不存在我们之前讨论的红外发散。她的论证如下: 在 z 三重态中，第三个元素 r 的存在将 p 限制在远离 $\sum$ 的区域，将 q 限制在远离 $\mathcal{H}$ 的区域；而在 l 三重态中，r 的作用相反，它将 p 限制在远离 $\mathcal{H}$ 的区域，将 q 限制在远离 $\sum$ 的区域。因此，对于这两种三重态，在更高维中不太可能积累任意数量的 $p-q$ 链接。

Let us note that Marr's qualitative argument regarding the would-be role played by r in killing off the IR divergences in higher dimensions does not guarantee the finiteness of $<\mathbf{H}>$ nor the emergence of the area law, for there could exist other less obvious and more subtle sources of divergences. Moreover, the finiteness of the result does not either guarantee that the resulting $<\mathbf{H}>$ will scale like the area. Therefore, the matter can only be settled by explicit calculation and this brings in technical difficulties that one has to deal with when considering higher dimensions and in particular $3+1$. These technical difficulties come from the necessity to evaluate the volumes needed to insure links and max/min conditions, which turned out to be complicated and in some cases intractable even in the flat case and for the link structure [17], let alone the triplet structure. However, there are indications that the case $2+1$ could be easier to handle analytically. Thus, it would be interesting to explicitly test the triplet proposals; in particular the z -triplet in $2+1$ is worth revisiting.

我们需要注意，马尔关于 r 在消除高维红外发散中作用的定性论证，不能保证 $<\mathbf{H}>$ 的有限性，也不能保证面积律的出现，因为可能存在其他更不明显、更微妙的发散来源。此外，结果的有限性也不能保证最终得到的 $<\mathbf{H}>$ 符合面积标度。因此，这个问题只能通过显式计算解决，而这在考虑更高维 (尤其是 3+1 维) 时会带来必须处理的技术难题。这些技术难题源于需要计算保证链接成立以及满足最大/最小条件所需的体积，结果表明该计算十分复杂，即使在平坦情形和链接结构下 [17] 某些情况也难以处理，更不用说三重态结构了。不过有迹象表明，2+1 维的情形更容易进行解析处理。因此，对三重态方案进行显式检验是很有意义的；尤其是 $2+1$ 中的 z 三重态，值得重新研究。

It is also worth mentioning that z - or l -triplets could work in $2+1$ but fail beyond this dimension, and one may be led to consider the "diamond" structure containing both types of triplet simultaneously.

还值得一提的是，z 三重态或 l 三重态可能在 $2+1$ 中适用，但在更高维失效，因此人们可能会考虑同时包含这两种三重态的“钻石”结构。

An Extended Notion of Horizon Molecules

视界分子的扩展概念

The definition of the horizon molecules as simple causal links crossing the horizon, supplemented with certain max/min conditions, worked nicely and gave promising results in 2-d reduced spacetimes, at least for null hypersurfaces. However, this success did not carry over to higher dimensions due to pathological IR divergences. The higher cardinality molecule definitions, namely, the triplets, proposed by Marr to cure these divergences turned out to be mathematically cumbersome and challenging beyond two dimensions, and so far no one has devised a technique that would allow analytical investigation of the triplet proposal in higher dimensions. This calculational impasse, the failure of the causal link proposal and the desire to extend the concept of horizon molecule to all causal horizons including black hole, acceleration, and cosmological horizons stimulated two recent sequential and related works by Barton et al. [20] and by Machet and Wang [21]. These two works will be the subject of the present section.

将视界分子定义为穿越视界的简单因果关联，并补充一定的最大/最小条件，这一定义在二维约化时空 (至少对零超曲面而言) 效果良好，得出了颇具前景的结果。然而，由于存在病态红外发散，这一成功无法推广到更高维度。马尔为解决这类发散提出了更高基数的分子定义，即三元组，但该定义在数学上十分繁琐，在二维以上维度的应用极具挑战，到目前为止，尚未有人提出可在高维对三元组方案进行解析研究的方法。这种计算上的僵局、因果关联方案的失败，以及将视界分子概念推广至包括黑洞视界、加速视界和宇宙学视界在内的所有因果视界的需求，推动了巴顿等人 [20] 与马谢和王 [21] 近期先后完成的两项相关工作。这两项工作就是本节的主题。

Our discussion will not cover the technical details presented in [20,21], but will be limited to introducing the key technical ideas, the results obtained, and their discussion.

我们的讨论不会涉及文献 [20,21] 中的技术细节，仅介绍核心技术思路、已得到的结果及相关讨论。

The Spacelike Hypersurface Case

类空超曲面情形

The extended proposal put forward by Barton et al. was devised to extend the notion of horizon molecule to more general causal horizons; thus, it does not refer to any particular black hole geometry and to produce the area law in higher dimensions.

Barton 等人提出的扩展方案旨在将视界分子的概念推广到更一般的因果视界；因此，它不局限于任何特定的黑洞几何，并且可以在更高维度中得到面积定律。

The definition goes as follows. Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime with a Cauchy surface $\sum$. Let $\mathcal{H}$ be a causal horizon, defined as the boundary of the past of a future inextendible timelike curve γ , i.e. $\mathcal{H} \ := \ \dot{I}^{-}(\gamma)$, and consider a causet $\mathcal{C}$ generated on $\mathcal{M}$ through a random sprinkling with a density ϱ_c .

定义如下。设 $(\mathcal{M}, g)$ 为具有柯西曲面 $\sum$ 的整体双曲时空，设 $\mathcal{H}$ 为因果视界，定义为未来不可延类时曲线 γ 的过去边界，即 $\mathcal{H} := \dot{I}^-(\gamma)$，现在考虑通过密度为 ϱ_c 的随机撒播在 $\mathcal{M}$ 上生成的因果集合 $\mathcal{C}$。

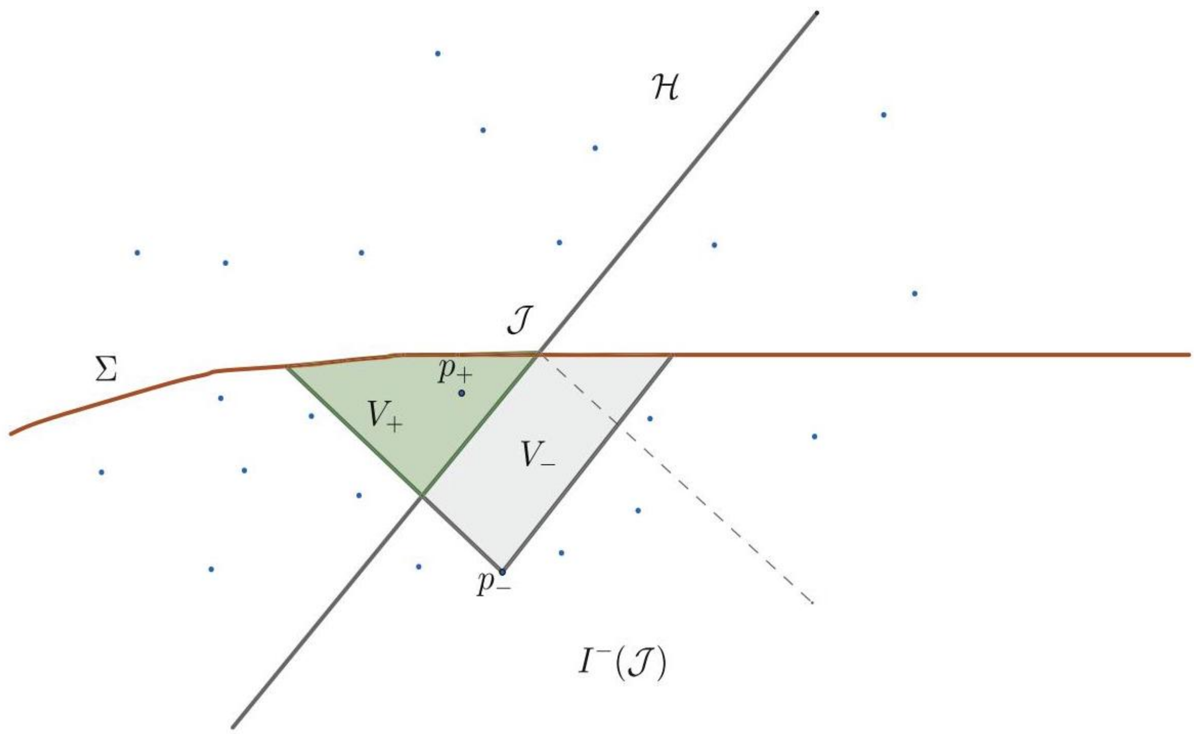


Fig. 8 A typical geometrical setting showing a typical horizon molecule (p_-, p_+) $(n = 1)$

图 8 展示典型视界分子 $(p_-, p_+)(n = 1)$ 的典型几何设置

Barton et al. proposal (2019) A horizon molecule with respect to spacelike hypersurface $\sum$ (a Cauchy surface) is a pair of elements of $\mathcal{C}, \{p_-, p_+\}$, such that

Barton 等人 2019 年的方案: 相对于类空超曲面 $\sum$ (柯西曲面) 的视界分子是 $\mathcal{C}, \{p_-, p_+\}$ 的一对元素, 满足

---

1. $p_- \prec p_+$
2. $p_- \in I^-(\sum) \cap I^-(\mathcal{H})$
3. $p_+ \in I^-(\sum) \cap I^+(\mathcal{H})$
4. p_+ is the only element in $I^-(\sum) \cap I^+(p_-)$

These conditions imply that the horizon molecule is a link. An illustration of this type of horizon molecules is depicted in Fig. 8.

这些条件说明视界分子是一个链接。这类视界分子的图示见图 8。

The above definition is easily seen to be generalizable to n -molecule $\{p_-, p_{1,+}, p_{2,+}, \cdots, p_{n,+}\}$ by requiring $p_- \prec p_{+,k}$ and $\{p_{1,+}, p_{2,+}, \cdots, p_{n,+}\}$ to be the only elements in $I^-(\sum) \cap I^+(p_-)$. However, our discussion of this proposal will be limited to the horizon molecule of minimal size $n = 1$. Actually the cardinality of the horizon molecules plays no essential technical role in the derivation of the results obtained in [20].

不难看出，上述定义可以通过要求 $p_- \prec p_{+,k}$ 和 $\{p_{1,+}, p_{2,+}, \cdots, p_{n,+}\}$ 是 $I^-(\sum) \cap I^+(p_-)$ 中仅有的元素，推广到 n 分子 $\{p_-, p_{1,+}, p_{2,+}, \cdots, p_{n,+}\}$。但本文对该方案的讨论将限于最小尺寸的视界分子 $n = 1$。实际上，在文献 [20] 的结果推导中，视界分子的基数并不发挥核心技术作用。

Before we move to the discussion of the derivation of the results of Barton et al., we find it instructive to compare the above definition with the original definition of horizon molecules as causal links with certain max/min conditions.

在我们讨论 Barton 等人的结果推导之前，将上述定义与满足特定最大/最小条件的因果链接形式的原初视界分子定义进行对比是有启发性的。

The extended horizon definition requires p_- to be a maximal element in $I^-(\sum) \cap I^-(\mathcal{H})$, or maximal-but-one in $I^-(\sum)$, and no similar maximality condition is imposed on p_+ . The requirement that p_- be maximal-but-one (or but- n) would, for instance, derive the expected number of horizon molecules directly to zero if the hypersurface was null (straight null plane) and the future of the horizon is unbounded, as it is the case for Rindler space, for example. For this and other reasons the null case has motivated an independent work by Machet and Wang [21], to which we will come lastly.

扩展的视界定义要求 p_- 是 $I^-(\sum) \cap I^-(\mathcal{H})$ 中的极大元，或是 $I^-(\sum)$ 中的次极大元，且不对 p_+ 施加类似的极大性条件。例如，如果超曲面是类光的 (平直零平面) 且视界的未来无界 (比如伦德勒空间就是这种情况)，那么要求 p_- 是次极大 (或第 n 大) 元会直接导致预期视界分子数为零。由于这个以及其他原因，零曲面情形推动了 Machet 和 Wang 的独立研究 [21]，我们最后再讨论这部分内容。

It was first shown in [20] that p_- is in the chronological past of $\mathcal{J} = \sum \cap \mathcal{H}$. Using similar steps we used to arrive to (6), it is not difficult to obtain the following integral representation for the expected number of such horizon molecules:

文献 [20] 首先证明了 p_- 位于 $\mathcal{J} = \sum \cap \mathcal{H}$ 的时序过去中。采用和我们得到式 (6) 类似的步骤，不难得到这类视界分子预期数的如下积分表示:

$$< \mathbf{H}_1 >= \varrho_c \int_{I^-(\mathcal{J})} \rho_c V_+(p) e^{-\varrho_c V(p)} dV_p \tag{26}$$

where

其中

$$V_+(p) := \operatorname{vol}\left(I^-\left(\sum\right) \cap I^+(\mathcal{H}) \cap I^+(p)\right), V_+(p) := \operatorname{vol}\left(I^-\left(\sum\right) \cap I^+(p)\right)$$

and p is used to denote the point p_+ . Figure 8 illustrates the volumes involved in the counting.

p 用来表示点 p_+ 。图 8 展示了计数涉及的体积。

The main result shown by Barton et al. is that under certain conceivable assumptions and in the continuum limit, the expected number of such defined horizon molecules, suitably rescaled, is equal to the area of $\mathcal{J}$, the intersection of $\sum$ and $\mathcal{H}$, up to a dimension-dependent constant of order one. Mathematically stated we have the following limit:

Barton 等人证明的主要结果是: 在某些合理假设下，连续极限中，经过适当重标度后，按上述定义的视界分子的预期数等于 $\mathcal{J}$ ($\sum$ 与 $\mathcal{H}$ 的交) 的面积，相差一个量级为 1、依赖维度的常数。数学上我们可以得到如下极限:

$$\lim_{\varrho_c \to \infty} \varrho_c^{\frac{2-d}{d}} < \mathbf{H}_1 >= a^{(d)} \int_{\mathcal{J}} dV_{\mathcal{J}} \tag{27}$$

where $dV_{\mathcal{J}}$ is the area measure on $\mathcal{J}$, and $a^{(d)}$ is a constant that only depends on the dimension d . Moreover, the approach to the limit involves finite ϱ_c corrections forming a derivative expansion of local geometric quantities on $\mathcal{J}$ and increasing powers of l_c , the discreteness length. In what follows we shall therefore explicitly keep track of ϱ_c and l_c .

其中 $dV_{\mathcal{J}}$ 是 $\mathcal{J}$ 上的面积测度，$a^{(d)}$ 是一个仅依赖于维度 d 的常数。此外，取极限的过程包含有限 ϱ_c 修正，这些修正构成了 $\mathcal{J}$ 上局部几何量的导数展开，且幂次随离散长度 l_c 递增。在下文中我们将明确追踪 ϱ_c 和 l_c 。

Rindler Horizon with a Flat Hypersurface

带平坦超曲面的林德勒视界

Before outlining the explicit calculation of [20], it would be instructive to present their heuristic argument supporting the validity of (27) in general. This heuristic argument actually summarizes the motivation behind defining the horizon molecules as such.

在概述文献 [20] 的显式计算之前，我们不妨先介绍作者们支持 (27) 普适成立的启发式论证。这一论证实际上总结了如此定义视界分子背后的动机。

Consider Fig. 8, the fact that p_- lies in $I^-(\mathcal{J})$ and is required to be maximal in this region means that p_- is close to $\mathcal{H}$, and as $\varrho_c \to \infty$ it gets closer. The requirement that p_- is maximal-but-one in $I^-(\sum) \cap I^+(\mathcal{H})$ pushes p_- towards $\sum$ and prevents it from moving to the past of $\mathcal{J}$.

参见图 8，p_- 位于 $I^-(\mathcal{J})$ 内且要求在该区域极大，这意味着 p_- 靠近 $\mathcal{H}$ ，且随着 $\varrho_c \to \infty$ 趋近，距离会越来越近。p_- 在 $I^-(\sum) \cap I^+(\mathcal{H})$ 内为次极大的要求将 p_- 推向 $\sum$ 方向，阻止它移动到 $\mathcal{J}$ 的过去一侧。

This tendency can be seen by inspecting the integrand of (27) in which the exponential will suppress any contribution from regions with $\varrho_c V(p) \gg 1$.

观察 (27) 的被积函数就能看出这一趋势: 被积函数中的指数项会压制来自含 $\varrho_c V(p) \gg 1$ 区域的所有贡献。

Contributions coming from points far from the horizon are suppressed by the $e^{-\varrho_c V_-(p)}$, whereas those close to the horizon but far from $\sum$ are suppressed by $e^{-\varrho_c V_+(p)}$. Therefore, the only region which gives a non-negligible contribution is a small and decreasing subregion of $I^-(\mathcal{J})$, immediately to the past of $\mathcal{J}$. This strongly suggests that in the limit, the integral will only depend on geometric quantities intrinsic to $\mathcal{J}$. On dimensional ground, the only geometric quantity that can appear on the RHS of (27) is the area of $\mathcal{J}$ times a dimensionless constant, $a^{(d)}$, which is independent of the geometry.

远离视界的点的贡献会被 $e^{-\varrho_c V_-(p)}$ 压制，而靠近视界但远离 $\sum$ 的点的贡献会被 $e^{-\varrho_c V_+(p)}$ 压制。因此，唯一能给出不可忽略贡献的区域是紧贴 $\mathcal{J}$ 过去一侧、属于 $I^-(\mathcal{J})$ 的一个不断缩小的小子区域。这有力地表明，在极限下积分仅依赖于 $\mathcal{J}$ 内禀的几何量。从量纲分析来看，能出现在 (27) 右侧的唯一几何量就是 $\mathcal{J}$ 的面积乘以无量纲常数 $a^{(d)}$ ，该常数与几何无关。

To prove (27) Barton et al. first proceeded by probing their definition on Minkowski d -dimensional spacetime, flat $\sum$, and Rindler horizon $\mathcal{H}$, in short all flat.

为证明 (27)，Barton 等人首先在闵氏 d 维时空、平坦 $\sum$ 和林德勒视界 $\mathcal{H}$ (简言之全为平坦情形) 下检验了他们的定义。

First, an inertial coordinate system is set up, $\left(x^0, x^1, y^\alpha\right), \alpha = 2, 3, \cdots d-1$, and the hypersurface $\sum$ is chosen at $x^0 = 0 . \mathcal{H}$ is given by $x^0 = -x^1$. For technical convenience a past-future swapped setup was instead used, so the domain of integration is $p \in I^+(\mathcal{J})$. The integrand is independent of y^α and the scaled expected number of horizon molecules takes the form

首先, 我们建立惯性坐标系 $\left(x^0, x^1, y^\alpha\right), \alpha = 2, 3, \cdots d-1$, 选取超曲面 $\sum$ 位于 $x^0 = 0 . \mathcal{H}$, 由 $x^0 = -x^1$ 给出。为方便技术处理，作者们改用了过去未来互换的设置，因此积分域为 $p \in I^+(\mathcal{J})$ 。被积函数与 y^α 无关，标度后的视界分子期望数形式为

$$\varrho_c^{\frac{2-d}{d}} < \mathbf{H}_1 >= \int_{\mathcal{J}} d^{d-2} y I^{d, \text{flat}}\left(l_c\right) \tag{28}$$

where $I^{d, \text{flat}}\left(l_c\right)$ is a dimensionless function given by

其中 $I^{d, \text{flat}}\left(l_c\right)$ 是由下式给出的无量纲函数

$$I^{(d, \text{flat})}\left(l_c\right) = l_c^{-(d+2)} \int_0^\infty dx^0 \int_{-x^0}^{x^0} dx^1 \widetilde{V}_+(x) e^{-\varrho_c \widetilde{V}(x)} \tag{29}$$

where $l_c = \varrho_c^{-1/d}$ is the discreteness length. In the limit $l_c \to 0$, the function $I^{(d, \text{flat})}\left(l_c\right)$ determines the constant $a^{(d)}$.

其中 $l_c = \varrho_c^{-1/d}$ 是离散长度，在极限 $l_c \to 0$ 下，函数 $I^{(d,\text{flat})}(l_c)$ 给出常数 $a^{(d)}$ 。

In this flat case $\widetilde{V}(x)$ is just the d -dimensional volume of a solid null cone of height x^0 . For $\widetilde{V}_+(x)$, it was not possible to derive a formula for general dimension d , but it was possible to compute it explicitly for the lower dimensions, $d = 2, 3$ and 4, and they are given respectively by

在这个平坦情形中，$\widetilde{V}(x)$ 恰好是高度为 x^0 的实心零锥的 d 维体积。对于 $\widetilde{V}_+(x)$ ，无法推导得到适用于任意维数 d 的公式，但可以对低维情况 $d = 2, 3$ 和 4 维显式计算，结果分别为

$$d = 2 : \widetilde{V}_+(x) = \frac{1}{4}\left(x^0 - x^1\right)^2$$

$$d = 3 : \widetilde{V}_+(x) = \frac{2}{3}\left(x^0\right)^3 \tan^{-1}\left(\sqrt{\frac{x^0 - x^1}{x^0 + x^1}}\right)$$

$$-\frac{1}{9}\left(2x^0 - x^1\right)\left(2x^0 + x^1\right)\sqrt{\left(x^0 - x^1\right)\left(x^0 + x^1\right)},$$

$$d = 4 : \widetilde{V}_+(x) = \frac{\pi}{48}\left(x^0 - x^1\right)^3\left(5x^0 + 3x^1\right).$$

A direct, but not straightforward, calculation leads to the following limits:

直接但非平凡的计算得到以下极限:

$$\lim_{l_c \to 0} I^{(2,\text{flat})}(l_c) = a^{(2)} = \frac{1}{3}$$

$$\lim_{l_c \to 0} I^{(3,\text{flat})}(l_c) = a^{(3)} = \frac{1}{4}\left(\frac{3}{\pi}\right)^{2/3}$$

$$\lim_{l_c \to 0} I^{(4,\text{flat})}(l_c) = a^{(4)} = \frac{\sqrt{3}}{10}.$$

Actually the constants $a^{(d)}$ were given in [20] for arbitrary n .

事实上，文献 [20] 已经针对任意 n 给出了常数 $a^{(d)}$ 。

It should be noted here that although Barton et al. computed the constants $a^{(d)}$ in the limit $l_c \to 0$ using Watson’s lemma, the function $I^{(d,\text{flat})}(l_c)$ is independent of the discreteness length l_c and equals $a^{(d)}$ at any discreteness scale. In other words we have

需要指出的是，尽管 Barton 等人利用沃森引理计算了 $l_c \to 0$ 极限下的常数 $a^{(d)}$ ，但函数 $I^{(d,\text{flat})}(l_c)$ 与离散长度 l_c 无关，在任意离散标度下都等于 $a^{(d)}$ 。换言之，我们有

$$I^{(d,\text{flat})}(l_c) := I^{(d,\text{flat})} = a^{(d)} = l_c^{-(d+2)} \int_0^\infty dx^0 \int_{-x^0}^{x^0} dx^1 V_+(x)\, e^{-\varrho_c V(x)}$$

(30)

and the above particular numerical values for $a^{(d)}$ are the exact values of the integrals $I^{(d,\,\text{flat})}(l_c)$ for different d regardless of the value of l_c .

且上述 $a^{(d)}$ 的特定数值就是不同 d 对应积分 $I^{(d,\,\text{flat})}(l_c)$ 的精确值，与 l_c 的取值无关。

There are indeed two ways to see why $I^{(d,\,\text{flat})}(l_c)$ must be independent of density of the sprinkling ϱ_c or l_c . The first is purely technical and based on a simple dimensional analysis of the integral (29). As $I^{(d,\,\text{flat})}(l_c)$ is dimensionless, and there is no length scale which can pair with l_c to form a dimensionless quantity, the result must be a pure number.

实际上有两种方式可以说明为何 $I^{(d,\,\text{flat})}(l_c)$ 一定与撒点密度 ϱ_c 或 l_c 无关。第一种是纯技术性的，基于对积分 (29) 的简单量纲分析。由于 $I^{(d,\,\text{flat})}(l_c)$ 是无量纲量，且不存在能与 l_c 结合构成无量纲量的长度标度，因此结果必然是一个纯数。

Another heuristic, but more intuitive, argument to understand why the derivation of the above area law should be independent of the density of the sprinkling in the all-flat setting is the following.

另一种理解为何全平背景下面积律的推导与撒点密度无关的启发式但更直观的论证如下。

In an all-flat setup, the two regions $I^-(\sum)\cap I^-(\mathcal{H})$ and $I^-(\sum)\cap I^+(\mathcal{H})$ are flat and infinite (unbounded). If one randomly sprinkles in points in both regions with a given density, say ϱ_1 , and considers another sprinkling with density ϱ_2 , both sprinklings should give the same result. The situation is just a matter of zoom in and zoom out, the trade-off here is simply that the number of molecules one loses by decreasing the density of sprinkling gains by moving further to the past (away from the intersection of the horizon and $\sum$). The density only tells us how far into the past we should go for the value of the constant $a^{(d)}$ to get effectively saturated, and in the very large density limit the molecules contributing to $<\mathbf{H}_1>$ are the ones located infinitesimally close to $\sum\cap\mathcal{H}$. In other words, if the integral over x^0 in (29) were cut off at some upper limit $\tau \gg l_c$, the value of the $l_c \to 0$ limit would not be affected, the deviation from the above limiting values tends to zero exponentially fast. This locality property, the fast exponential vanishing of the difference, will be crucial for the discussion of the general curvature case to which we now turn.

在全平设置中，两个区域 $I^-(\sum)\cap I^-(\mathcal{H})$ 和 $I^-(\sum)\cap I^+(\mathcal{H})$ 都是平坦且无限 (无界) 的。如果我们以给定密度 (例如 ϱ_1) 在两个区域中随机撒点，再换密度 ϱ_2 做一次撒点，两次得到的结果应当一致。这一情况本质只是缩放的问题: 降低撒点密度损失的灰洞分子数，完全可以通过向更早的过去 (远离视界与 $\sum$ 的交点) 移动得到补充。密度仅告诉我们，要让常数 $a^{(d)}$ 的取值有效饱和，需要向过去延伸多远；在极大密度极限下，对 $<\mathbf{H}_1>$ 有贡献的分子都是位于 $\sum\cap\mathcal{H}$ 无穷近处的分子。换言之，如果将 (29) 中对 x^0 的积分在某个上限 $\tau \gg l_c$ 处截断，也不会影响 $l_c \to 0$ 极限的取值，与上述极限值的偏差会按指数速率快速趋于零。这种局域性，即偏差的指数快速衰减，对我们接下来要讨论的一般曲率情形至关重要。

The General Curvature Case

一般曲率情形

Our discussion of the general curvature case will more or less be sketchy, escaping technical detail and just highlighting the crucial steps of the calculation of Barton et al.

我们对一般曲率情形的讨论或多或少会是梗概性的，略去技术细节，仅强调 Barton 等人计算中的关键步骤。

The key elements in proving the limit (26) in the general curvature setting were first the construction of Florides-Synge normal coordinates (FSNCs) based on the co-dimension 2 spacelike submanifold $\mathcal{J}$ and second the locality argument. Such coordinate system construction is always possible in tubular neighborhood about a submanifold of any co-dimension in any Riemannian or pseudo-Riemannian manifold [27].

在一般曲率背景下证明极限 (26) 的核心要素，首先是基于余维数为 2 的类空子流形 $\mathcal{J}$ 构造 Florides-Synge 正规坐标 (FSNC)，其次是局域性论证。这种坐标构造在任意黎曼或伪黎曼流形中任意余维数子流形的管状邻域内总是可行的 [27]。

For $d > 2$, let $z^a = \left(x^A, y^\alpha\right)(A = 0, 1, \alpha = 2, \cdots, d-1)$ denote the FSNCs constructed within a small enough tubular neighborhood $\mathcal{N}$ about $\mathcal{J}$. For $d = 2$ FSBCs are just the Riemann normal coordinates based on the intersection point $\mathcal{J}$.

对于 $d > 2$, 设 $z^a = \left(x^A, y^\alpha\right)(A = 0, 1, \alpha = 2, \cdots, d-1)$ 为在 $\mathcal{J}$ 足够小的管状邻域 $\mathcal{N}$ 内构造的 FSNC。对于 $d = 2$ ， FSBC 正是基于交点 $\mathcal{J}$ 的黎曼正规坐标。

The next step is to assume the existence of a length scale τ such that $l_c \ll \tau \ll L_G$, where L_G is the smallest geometric scale in the setup. This assumption is reasonable, because the continuum approximation of causal set is only valid when the curvature length scales involved in the problem are much larger than the discreteness scale l_c .

下一步是假设存在长度标度 τ 满足 $l_c \ll \tau \ll L_G$ ， 其中 L_G 是该框架下最小的几何标度。这个假设是合理的，因为因果集合的连续近似仅在问题涉及的曲率长度标度远大于离散标度 l_c 时成立。

Consider now the region $\mathcal{R}_\tau$ defined as

现在考虑区域 $\mathcal{R}_\tau$ ， 其定义为

$$\mathcal{R}_\tau := \left\{p \in I^-(\mathcal{J}) \cap \mathcal{N} : -\tau < x^0(p) < 0\right\}$$

where τ is assumed to be small enough that this region is inside the tubular neighborhood $\mathcal{N}$.

其中假设 τ 足够小，使得该区域完全包含在管状邻域 $\mathcal{N}$ 内。

Let $\overline{\mathcal{R}}_\tau := I^-(\mathcal{N}) \setminus \mathcal{R}_\tau$ denote the complement of $\mathcal{R}_\tau$; then the integral (27) naturally splits into a part over $\mathcal{R}_\tau$ and another over $\overline{\mathcal{R}}_\tau$.

设 $\overline{\mathcal{R}}_\tau := I^-(\mathcal{N}) \setminus \mathcal{R}_\tau$ 为 $\mathcal{R}_\tau$ 的补集，则积分 (27) 自然拆分为 $\mathcal{R}_\tau$ 上的部分和 $\overline{\mathcal{R}}_\tau$ 上的另一部分。

Now, in [20] it was argued, using the locality argument, that the integral over $\overline{\mathcal{R}}_\tau$ tends to zero faster than any power of l_c , actually exponentially suppressed, and hence, its contribution can be ignored. Therefore, the surviving part of the expected value can be written as a local integral over $\mathcal{R}_\tau$:

文献 [20] 中借助局域性论证指出，$\overline{\mathcal{R}}_\tau$ 上的积分比 l_c 的任意幂次都更快趋于零，实际上是指数抑制的，因此其贡献可以忽略。由此，期望值的剩余部分可以写为 $\mathcal{R}_\tau$ 上的局域积分:

$$\varrho_c^{\frac{2-d}{d}} < \mathbf{H}_1 >= \varrho_c^{\frac{2-d}{d}+2} \int_{\mathcal{R}_\tau} V_+ (p)\, e^{-\varrho_c V(p)} dV_p. \tag{31}$$

In view of the fact that the region $\mathcal{R}_\tau$ lies by choice within the tubular neighborhood, the constructed FSNCs can be used to express the expectation value explicitly as

根据构造，区域 $\mathcal{R}_\tau$ 位于管状邻域内，因此可以利用构造好的 FSNC 将期望值显式表示为

$$\varrho_c^{\frac{2-d}{d}} < \mathbf{H}_1 >= \varrho_c^{\frac{2-d}{d}+2} \int_{\mathcal{J}} d^{d-2}y \int_{-\tau}^{0} dx^0 \int_{x^0}^{-x^0} dx^1 \sqrt{-g(x,y)} V_+ (x,y)\, e^{-\varrho_c V(x,y)}$$

(32)

where $g(x,y)$ is the determinant of the metric.

其中 $g(x,y)$ 是度量的行列式。

Let $\sigma_{\alpha\beta}$ denote the induced metric on $\mathcal{J}$, and then (31) can be written as

设 $\sigma_{\alpha\beta}$ 为 $\mathcal{J}$ 上的诱导度量，则 (31) 可以写为

$$\varrho_c^{\frac{2-d}{d}} < \mathbf{H}_1 >= \int_{\mathcal{J}} d^{d-2}y \sqrt{-\sigma(y)} I^{(d)}(y; l_c, \tau) + \cdots \tag{33}$$

where $I^{(d)}(y; l_c, \tau)$ is defined by

其中 $I^{(d)}(y; l_c, \tau)$ 定义为

$$I^{(d)}(y; l_c, \tau) \;:=\; l_c^{-(d+2)} \int_{-\tau}^{0} dx^0 \int_{x^0}^{-x^0} dx^1 \sqrt{\frac{-g(x,y)}{\sigma(y)}} V_+ (x,y)\, e^{-\varrho_c V(x,y)}$$

(34)where $\sigma(y)$ is the determinant of the induced metric.

其中 $\sigma(y)$ 是诱导度量的行列式。

The factor $\sqrt{\frac{-g(x,y)}{\sigma(y)}}$ makes $I^{(d)}(y; l_c, \tau)$ a scalar on $\mathcal{J}$ and can be rewritten in a free coordinate notation as $I^{(d)}(q; l_c, \tau), q \in \mathcal{J}$.

因子 $\sqrt{\frac{-g(x,y)}{\sigma(y)}}$ 使得 $I^{(d)}(y; l_c, \tau)$ 成为 $\mathcal{J}$ 上的标量，可以在自由坐标记号下重写为 $I^{(d)}(q; l_c, \tau), q \in \mathcal{J}$ 。

The next crucial step in the calculation of Barton et al. is to show that $I^{(d)}(q; l_c, \tau)$ admits the following local expansion:

Barton 等人计算中下一步关键步骤是证明 $I^{(d)}(q; l_c, \tau)$ 容许如下局域展开:

$$I^{(d)}(q; l_c, \tau) = a^{(d)} + l_c \sum_i b_i^{(d)} \mathcal{G}_i(q) + O\left(l_c^2\right) \tag{35}$$

where $a^{(d)}$ and $b_i^{(d)}$ are constants that only depend upon the dimension d . For instance, $a^{(d)}$ is the same constant obtained in the flat case. $\mathcal{G}_i(q)$ is the largest set of mutually independent geometric scalars of length dimension L^{-1} , like the extrinsic curvature K or the null expansion θ evaluated at q .

其中 $a^{(d)}$ 和 $b_i^{(d)}$ 是仅依赖于维度 d 的常数。例如，$a^{(d)}$ 就是平直情形下得到的同一个常数。$\mathcal{G}_i(q)$ 是长度量纲为 L^{-1} 的最大互独立几何标量集合，比如计算在 q 处的外曲率 K 和类光膨胀标量 θ 。

Again, switching to an order-reversed setup, $I^{(d)}(q; l_c, \tau)$ is written as

再次说明，转换到逆序设置后，$I^{(d)}(q; l_c, \tau)$ 可写为

$$I^{(d)}(y; l_c, \tau) = l_c^{-(d+2)} \int_0^\tau dx^0 \int_{-x^0}^{x^0} dx^1 \sqrt{\frac{-g(x,y)}{\sigma(y)}} V_+(x,y) e^{-\varrho_c V(x,y)}. \tag{36}$$

At this stage one is free to choose any coordinates on $\mathcal{J}$, and a suitable choice is RNC’ s y^α centered about $q \in \mathcal{J}; y^\alpha(a) = 0$. As all expressions appearing in (36) are evaluated at $y^\alpha(a) = 0$, the argument y will be dropped entirely to write

此时我们可以任意选择 $\mathcal{J}$ 上的坐标系，合适的选择是以 $q \in \mathcal{J}; y^\alpha(a) = 0$ 为中心的 RNC 坐标系 y^α 。由于 (36) 中出现的所有表达式都在 $y^\alpha(a) = 0$ 处求值，因此我们将完全省略自变量 y ，写作

$$I^{(d)}(q; l_c, \tau) = l_c^{-(d+2)} \int_0^\tau dx^0 \int_{-x^0}^{x^0} dx^1 \sqrt{-g(x)} V_+(x) e^{-\varrho_c V(x)}. \tag{37}$$

Note that $\sigma(0) = 1$ in these RNCs on $\mathcal{J}, \sigma_{\alpha\beta}(0) = \delta_{\alpha\beta}$.

请注意，$\sigma(0) = 1$ 处于 $\mathcal{J}, \sigma_{\alpha\beta}(0) = \delta_{\alpha\beta}$ 上的这些 RNC 坐标系中。

Now spacetime RNCs $Z^a = \left(X^A, Y^\alpha\right)$ can be introduced within a neighborhood $\mathcal{U}$ about q , such that $X^A = x^A$ and such that the coordinate vectors $\frac{\partial}{\partial Y^\alpha} = \frac{\partial}{\partial v^\alpha}$ at q . With this choice the determinant of the metric, evaluated at q , keeps the same form in terms of the coordinates x^A and X^A , and we have

现在可以在围绕 q 的邻域 $\mathcal{U}$ 内引入时空 RNC 坐标系 $Z^a = \left(X^A, Y^\alpha\right)$ ，满足 $X^A = x^A$ ，且坐标向量满足在 q 处的条件 $\frac{\partial}{\partial Y^\alpha} = \frac{\partial}{\partial v^\alpha}$ 。通过该选择，在 q 处求值的度规行列式仍可通过坐标 x^A 和 X^A 保持相同形式，我们得到

$$I^{(d)}(q; l_c, \tau) = l_c^{-(d+2)} \int_0^\tau dX^0 \int_{-X^0}^{X^0} dX^1 \sqrt{-g(X)} V_+(X) e^{-\varrho_c V(X)}. \tag{38}$$

The determinant $g(X)$ can be expanded in small X^A relative to the curvature scales of spacetime at q :

行列式 $g(X)$ 可以相对于 q 处时空的曲率尺度，按小量 X^A 展开:

$$\sqrt{=g(X)} = 1 - \frac{1}{6} R_{AB} X^A X^B + O\left(Z^3\right) \tag{39}$$

where R_{AB} is the Ricci tensor with indices restricted to $A, B = 0, 1$. To bring out the role of the different length scales of the problem, the smallest length scale L_G is used to define a dimensionless tensor $\widehat{R}_{ab} := L_G^2 R_{ab}$, and τ is used to reexpress the above expansion in terms of dimensionless coordinates $\widehat{Z}_a := Z^a/\tau$:

其中 R_{AB} 是指标限制在 $A, B = 0, 1$ 上的里奇张量。为了凸显问题中不同长度尺度的作用，我们使用最小长度尺度 L_G 定义无量纲张量 $\widehat{R}_{ab} := L_G^2 R_{ab}$ ，并通过 τ 将上述展开改写为无量纲坐标 $\widehat{Z}_a := Z^a/\tau$ 的形式:

$$\begin{aligned} \sqrt{-g(X)} &= 1 - \frac{1}{6}\left(\frac{\tau}{L_G}\right)^2 \widehat{R}_{AB} \widehat{X}^A \widehat{X}^B + O\left(Z^3\right) \\ &= 1 - \frac{1}{6} \varepsilon^2 \widehat{R}_{AB} \widehat{X}^A \widehat{X}^B + O\left(\varepsilon^3\right). \end{aligned} \tag{40}$$

In view of the fact that $\varepsilon = \tau/L_G \ll 1$, and L_G is the smallest geometric scale, the correction $\frac{1}{6} R_{AB} X^A X^B$ is of order ε^2 .

鉴于 $\varepsilon = \tau/L_G \ll 1$ ，且 L_G 是最小几何尺度，修正项 $\frac{1}{6} R_{AB} X^A X^B$ 的阶为 ε^2 。

The volumes $V(X)$ and $V_+(X)$ can similarly be expanded around the flat ones in the neighborhood $\mathcal{U}$. Using different explicit geometric setups, in particular different choices for the hypersurface $\sum$, Barton et al. suggested the following general expansion for the volumes

体积 $V(X)$ 和 $V_+(X)$ 同样可以在邻域 $\mathcal{U}$ 内围绕平直体积展开。通过不同的显式几何设置，尤其是对超曲面 $\sum$ 的不同选择，Barton 等人提出了体积的如下通用展开式

$$\begin{aligned} V\left(X_p\right) &= \widetilde{V}\left(X_p\right)\left[1 + \sum_i \mathcal{G}_i(q) f_i\left(X_p\right) + O\left(\varepsilon^2\right)\right] \\ V\left(X_p\right)_+ &= \widetilde{V}_+\left(X_p\right)\left[1 + \sum_i \mathcal{G}_i(q) f_{+,i}\left(X_p\right) + O\left(\varepsilon^2\right)\right] \end{aligned} \tag{41}$$

where $\widetilde{V}\left(X_p\right)$ and $\widetilde{V}_+\left(X_p\right)$ are the volumes from the all-flat case discussed previously. $f_i\left(X_p\right)$ and $f_{+,i}\left(X_p\right)$ are functions of length dimension L .

其中 $\widetilde{V}\left(X_p\right)$ 和 $\widetilde{V}_+\left(X_p\right)$ 是前文讨论的全平直情形下的体积，$f_i\left(X_p\right)$ 和 $f_{+,i}\left(X_p\right)$ 是长度量纲为 L 的函数。

Using equations (40) and (41) the following expansion for $I^{(d)}(q; l_c, \tau)$, it is easy to obtain

利用方程 (40) 和 (41)，容易得到 $I^{(d)}(q; l_c, \tau)$ 的如下展开式

$$
\begin{aligned}
I^{(d)}(q;l_c,\tau) = l^{-(d+2)}\int_0^{\tau} dX^0 e^{-\rho_c \widetilde{V}(X^0)} \Bigg\{ \int_{-X^0}^{X^0} dX^1 \widetilde{V}_+(X) \\
+\sum_i \mathcal{G}_i(q) \Bigg[\int_{-X^0}^{X^0} dX^1 \widetilde{V}_+(X) f_{+,i}(X) \\
-\rho_c \widetilde{V}(X^0) \int_{-X^0}^{X^0} dX^1 \widetilde{V}_+(X) f_i(X) \Bigg] + O(\varepsilon^2) \Bigg\}
\end{aligned}
\tag{42}
$$

where the fact that the flat cone volume $\widetilde{V}$ only depends on X^0 was used, and the subscript p from the coordinates X^A has been removed.

这里用到了平锥体积 $\widetilde{V}$ 仅依赖于 X^0 这一性质，并且已去掉坐标 X^A 的下标 p

The integral in the first line is just the flat contribution $I^{(d,\text{ flat })(l)}$ given by (36) up to a difference which vanishes exponentially fast in the limit $l_c \to 0$.

第一行中的积分就是式 (36) 给出的平坦贡献 $I^{(d,\text{ flat })(l)}$，仅相差一项在极限 $l_c \to 0$ 下指数快速衰减的差值

By dimensional argument and using Watson's lemma again, the expression in square bracket of (42) can be shown to evaluate to a term of the form Cl_c , for some constant C , as $l_c \to 0$. Similarly, the $O(\varepsilon^2)$ corrections tend to a function of order $O(l^2)$. Therefore, the expansion of (35) follows.

通过量纲分析并再次利用沃森引理，可以证明当 $l_c \to 0$ 时，式 (42) 中括号内的表达式可化为 Cl_c 形式的项，其中 C 为某个常数；同理，$O(\varepsilon^2)$ 修正项趋近于一个 $O(l^2)$ 阶的函数。因此可得式 (35) 的展开式。

The constants $a^{(d)}$ are given by their flat values, $I^{(d,\text{ flat })} = a^{(d)}$. The explicit form of the constants $b_i^{(d)}$ can be determined once a geometric setup is chosen. For instance, Barton et al. have explicitly evaluated these constants for two different geometric setups [20].

常数 $a^{(d)}$ 由其平坦值给出，即 $I^{(d,\text{ flat })} = a^{(d)}$ 。选定几何结构后即可确定常数 $b_i^{(d)}$ 的具体形式。例如，Barton 等人已在两种不同几何结构下计算得到了这些常数的具体结果 [20]。

The Null Hypersurface Case

类零超曲面情形

The horizon molecule proposal of Barton et al. was specially devised to work when hypersurfaces of spacelike nature are considered. However, and as we mentioned earlier, there are good reasons for requiring any horizon molecule definition to be also valid in the case of null hypersurfaces. This issue was not raised nor discussed in [20], but a subsequent recent work by Machet and Wang addressed this question and investigated in detail the extension of this definition to encompass null hypersurfaces intersecting the horizon. The goal of the following subsection is to give a concise report of the main results and conclusions of Machet and Wang.

Barton 等人的视界分子方案最初是专门为类空超曲面设计的。但正如我们前文提到，有充分的理由要求任意视界分子定义也适用于类零超曲面。文献 [20] 并未提出或讨论这一问题，随后 Machet 与 Wang 的近期研究针对该问题展开工作，详细探究了将该定义推广至与视界相交的类零超曲面的方案。下文小节的目标是简要介绍 Machet 与 Wang 的主要结果与结论。

Let us first give a general look at the problem to see how the success of Barton et al.'s proposal is tied to the spacelike nature of the hypersurface.

我们首先对该问题做一个整体梳理，看看 Barton 等人方案的成立性是如何与超曲面的类空性质绑定的。

To that end consider the all-flat case of Fig. 9, a Rindler horizon in Minkowski space, with $\sum$ being a straight null plane. As can easily be seen, the region $I^{-}(\sum) \cap I^{+}(\mathcal{H}) \cap I^{+}(p_{-})$ is unbounded with infinite volume for any randomly selected point p_{-}, and the expected number of such horizon molecules is thus directly derived to zero. Therefore, before any sensible calculation of the expected number is started, one has to first bound this domain, at least for the all-flat case. This can be done by either considering a folded null plane instead of the straight one, or by taking a null hypersurface with different shapes like a downward light cone. These two configurations were probed in [21] to compute the expected number of horizon molecules, although the motivation there for bounding the region $I^{-}(\sum) \cap I^{+}(\mathcal{H}) \cap I^{+}(p_{-})$ was to avoid any potential IR divergence.

为此我们考虑图 9 的全平直情形: 闵氏空间中的林德勒视界，其中 $\sum$ 是一个平直零平面。不难发现，对任意任意选取的点 p_{-}，区域 $I^{-}(\sum) \cap I^{+}(\mathcal{H}) \cap I^{+}(p_{-})$ 都是无界的，且体积无穷大，因此这类视界分子的期望数直接为零。因此，在开展任何合理的期望数计算之前，必须首先对该区域定界，至少在全平直情形中需要如此。定界可以通过两种方式实现: 改用折叠零平面替代平直零平面，或是采用其他形状的类零超曲面，例如下行光锥。文献 [21] 对这两种构造都进行了探究，以计算视界分子的期望数，不过其中对区域 $I^{-}(\sum) \cap I^{+}(\mathcal{H}) \cap I^{+}(p_{-})$ 定界的出发点是避免潜在的红外发散。

Let us note that, as we discussed in section "The Spacelike Hypersurface in 2-D Reduced", one cannot approach the null case by invoking a continuity argument, like the one given by equation (21), by continuously deforming the spacelike result to obtain that of a null hypersurface. Therefore, the issue can only be settled by explicit calculation.

需要注意的是，正如我们在“二维约化下的类空超曲面”一节讨论的，我们无法通过连续性论证得到类零情形的结果——即无法像式 (21) 那样，通过连续形变类空情形的结果得到类零超曲面的结论。因此该问题只能通过显式计算解决。

Rindler Horizon in Minkowski Spacetime

闵氏时空中的林德勒视界

Machet and Wang applied Barton et al.'s definition first to a Rindler horizon in Minkowski space. As we have already mentioned, in the all-flat case and due to the unboundedness of the region $I^{-}(\sum) \cap I^{+}(\mathcal{H}) \cap I^{+}(p_{-})$, the expected number of Barton et al. horizon molecules is identically zero; therefore, for any sensible calculation to get started with such definition and geometric setup, one has first to introduce some IR regulator to bound the domain $I^{-}(\sum)\cap I^{+}(\mathcal{H})\cap I^{+}(p_{-})$. This can be achieved, for instance, by taking $\sum$ to be a folded null plane or a downward light cone.

Machet 与 Wang 首先将 Barton 等人的定义应用于闵氏空间中的林德勒视界。正如前文所述, 在全平坦情形下, 由于区域 $I^{-}(\sum)\cap I^{+}(\mathcal{H})\cap I^{+}(p_{-})$ 无界, Barton 等人视界分子的期望数恒为零; 因此, 要使用该定义和该几何结构开展任何合理计算，必须先引入红外调节器对区域 $I^{-}(\sum)\cap I^{+}(\mathcal{H})\cap I^{+}(p_{-})$ 做边界限制。这一点可以通过例如将 $\sum$ 取为折叠类光平面或向下光锥来实现。

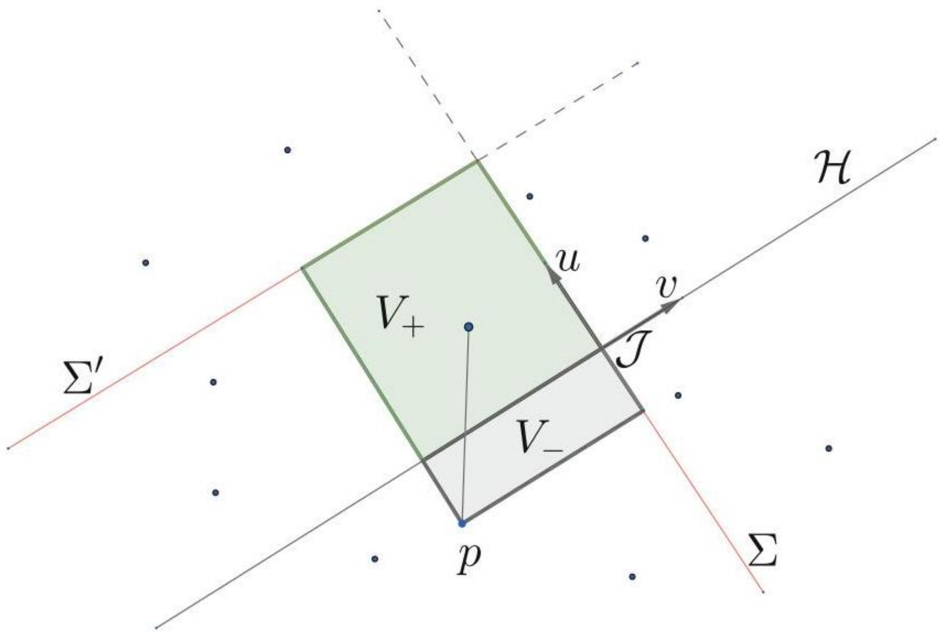


Fig. 9 An all-flat setting: a folded null plane crossing in a Rindler horizon

图 9 全平坦设置: 折叠类光平面与林德勒视界相交

The setup of a folded null plane is depicted in Fig.9. A global coordinate (v,u,y^{α}) is set up; the null hypersurface $\sum$ is the u -axis, $v=0$; and the horizon $\mathcal{H}$ is the v -axis, $u=0$. Another null hypersurface $\sum'$ is given by $u=\lambda$, with $\lambda>0$. The union $\sum'\cup\sum$ is the folded null plane with respect to which the expected number of horizon molecules is to be counted.

折叠类光平面的设置如图 9 所示。我们建立了全局坐标 (v,u,y^{α}) ; 类光超曲面 $\sum$ 是 u 轴，即 $v=0$; 视界 $\mathcal{H}$ 是 v 轴，即 $u=0$ 。另一类光超曲面 $\sum'$ 由 $u=\lambda$ 给出，满足 $\lambda>0$ 。并集 $\sum'\cup\sum$ 即为折叠类光平面，我们需要对该平面上的视界分子期望数进行计数。

In this setup the volumes $V_{+}(p)$ and $V(p)$ can explicitly be computed for an arbitrary dimension and are given by

在该设置中，体积 $V_{+}(p)$ 和 $V(p)$ 可对任意维数显式计算，结果为

$$V_{f}(u,v,\lambda)=\alpha_{d}(2v(u+\lambda))^{d/2},$$

$$V_{+,f}(u,v,\lambda)=\alpha_d(2v)^{d/2}\left((u+\lambda)^{d/2}-u^{d/2}\right)$$

where α_d is a constant dependent on the dimension.

其中 α_d 是依赖于维数的常数。

It follows again that the expected number of horizon molecules can be written as

由此可得，视界分子的期望数可写为

$$\varrho_c^{\frac{2-d}{d}}<\mathbf{H}_1>=\int_{\mathcal{J}}d^{d-2}yI_{\text{null}}^{(d,flat)}(l_c,\lambda) \tag{43}$$

where $I_{\text{null}}^{(d,\text{ flat})}(l_c,\lambda)$ is given by

其中 $I_{\text{null}}^{(d,\text{ flat})}(l_c,\lambda)$ 由下式给出

$$I_{\text{null}}^{(d,\text{ flat})}(l_c,\lambda)=l_c^{-(d+2)}\int_0^\infty dv\int_0^\infty duV_{+,f}(u,v,\lambda)\,e^{-\rho_cV_f}. \tag{44}$$

Using the explicit formulas for $V_{+,f}$ and V_f , one gets

利用 $V_{+,f}$ 和 V_f 的显式公式，可得

$$I_{\text{null}}^{(d,\text{ flat})}(l_c,\lambda)=\frac{\alpha_d^{-2/d}}{d}\Gamma[2/d+1]\int_0^\infty du\left((u+\lambda)^{d/2}-u^{d/2}\right)(u+\lambda)^{-1-d/2}.$$

(45)

It is noticeable that any dependence on the discreteness scale has disappeared and, therefore, by dimensional analysis, $I_{null}^{(d,flat)}(l_c,\lambda)$ should be independent of λ . The above integral can be evaluated and one obtains

值得注意的是，对离散标度的所有依赖都已消失，因此通过量纲分析可知，$I_{null}^{(d,flat)}(l_c,\lambda)$ 应当与 λ 无关。对上述积分求值后可得

$$I_{\text{null}}^{(d,\text{ flat})}(l_c,\lambda)\;:=\;I_{\text{null}}^{(d,\text{ flat})}=a_{\text{null}}^{(d,\text{ flat})}=\frac{\alpha_d^{-2/d}}{d}\Gamma[2/d+1]\,H_{d/2} \tag{46}$$

where H_k is the k th harmonic number.

其中 H_k 是第 k 个调和数。

Actually, Machet and Wang also carried out the calculation for the n -molecule and obtained a formula which can be exactly evaluated for each n .

事实上，Machet 和 Wang 还对 n 分子做了计算，得到了一个对任意 n 都可以精确求值的公式。

It follows then that

因此可得

$$\varrho_c^{\frac{2-d}{d}} < \mathbf{H}_1 >= a_{\text{null}}^{(d,\,\text{flat})} \int_{\mathcal{J}_t} dV_{\mathcal{J}} \tag{47}$$

Some comments on this result are in order.

我们在此对该结果做一些说明。

It is first interesting to note that the resulting constants $a_{\text{null}}^{(d,\,\text{flat})}$ are different from the constants $a^{(d,\,\text{flat})}$ obtained in the spacelike case, as can be checked by substituting for particular values of d .

首先值得注意的是，最终得到的常数 $a_{\text{null}}^{(d,\,\text{flat})}$ 与类空情形下得到的常数 $a^{(d,\,\text{flat})}$ 不同，代入 d 的特定值即可验证这一点。

Moreover, the final result is independent of the position of the $\sum'$, the parameter λ . Therefore, one may take the IR regulator to infinity without changing the result, hence going back to null plane case, and this goes in contradiction with the fact that if one started with a null plane, the expected number of horizon molecules would be identically zero. Again, we see that this horizon molecule counting is sensitive to how some limits are taken.

此外，最终结果与 $\sum'$ 的位置即参数 λ 无关。因此我们可以将红外调节器取到无穷而不改变结果，从而回到零平面的情况；这一结论与下述事实矛盾: 如果从一开始就采用零平面，视界分子的期望数会恒等于零。我们再次看到，这种视界分子计数对取极限的方式敏感。

The independence of $a^{(d,flat)}$ from l_c is actually related to its independence of λ and the flat setup is used to do the counting. As it was pointed out in [21], because of Lorentz invariance of the counting and the fact that one can always boost the system in the u direction to pull the surface $\sum'$ arbitrarily close to $\mathcal{H}$, the result should not depend on λ . If $\mathcal{J}$ has no geometrical quantity associated to it, e.g., intrinsic curvature, then l_c has no length scale to couple with, and therefore, $a^{(d,\,\text{flat})}$ can only be a pure number. Similarly to the spacelike case, to establish the area law in the all-flat setup using this counting, the continuum limit plays no role, and equation (47) is valid for any finite ϱ_c .

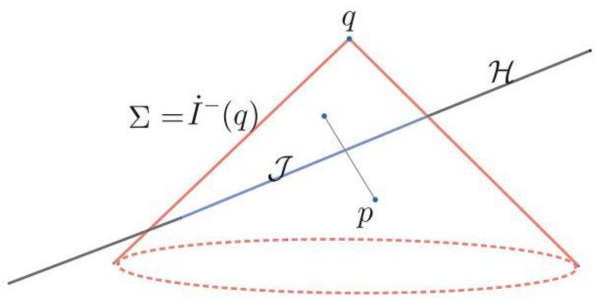
$a^{(d,flat)}$ 对 l_c 的无关性实际上和它对 λ 的无关性有关，我们利用平直背景来完成计数。正如文献 [21] 指出的，由于计数具有洛伦兹不变性，且总能通过洛伦兹 boost 将系统沿 u 方向移动，使曲面 $\sum'$ 任意接近 $\mathcal{H}$ ，因此结果不应当依赖于 λ 。若 $\mathcal{J}$ 不带有与之关联的几何量 (例如内禀曲率)，则 l_c 不存在可耦合的长度标度，因此 $a^{(d,\,\text{flat})}$ 只能是一个纯数。与类空情形类似，若要在此全平直背景中利用该计数建立面积定律，连续极限不起作用，且方程 (47) 对任意有限的 ϱ_c 都成立。

Another setup probed in [21] was again a Rindler horizon in Minkowski space but with a null hypersurface $\sum$ having a different shape, namely, a downward light cone.

文献 [21] 探究的另一个设定仍是闵氏时空的林德勒视界，但其中的类光超曲面 $\sum$ 具有不同形状，即向下的光锥。

Fig. 10 A downward light cone intersecting a Rindler horizon

图 10 与林德勒视界相交的向下光锥



The downward light cone $\sum$ is defined to be the boundary of the causal past of a point $q \in \mathcal{H}$, i.e., $\dot{I}^-(q)$, Fig. 10.

向下光锥 $\sum$ 被定义为点 $q \in \mathcal{H}$ 的因果过去的边界，即 $\dot{I}^-(q)$ ，见图 10。

The volumes V_+ and $V(p)$ are now given by

体积 V_+ 和 $V(p)$ 现在可表示为

$$V_{+,q}+(p) := \mathrm{vol}\left(I(p,q) \cap I^+(\mathcal{H})\right), \; V_q(p) := \mathrm{vol}\left(I(p,q)\right).$$

Machet and Wang could compute these volumes explicitly in $d = 4$, in terms of the null coordinates of p and the affine distance between the horizon and the point q and obtained a general formula for $I_{\text{cone}}^{(4,\,\text{flat})}$ which reduces for $n = 1$ to

Machet 和 Wang 在 $d = 4$ 中利用 p 的类光坐标以及视界和点 q 之间的仿射距离，对这些体积进行了显式计算，得到了 $I_{\text{cone}}^{(4,\,\text{flat})}$ 的通式，该通式在 $n = 1$ 下约化为

$$I_{\text{cone}}^{(4,\,\text{flat})} \frac{1}{12}\sqrt{\frac{3}{2}}\left(\frac{27}{4} - {}_2F_1\left(1,-1,-2;\frac{3}{2}\right)\right) = a^{(4)}. \tag{48}$$

We notice that $I_{\text{cone}}^{(4,\,\text{flat})}$ turns out to be again independent of the discreteness scale and of the other length scale provided by the affine distance between the horizon and the point q . The explanation of this independence is similar to the folded null plane.

我们注意到，$I_{\text{cone}}^{(4,\,\text{flat})}$ 结果再次不依赖于离散标度，也不依赖于视界与点 q 之间仿射距离带来的其他长度标度。这种无关性的解释和折叠类光平面情形类似。

It is noteworthy that here too the constant $a^{(4)}$ is different from its counterpart in the spacelike hypersurface.

值得注意的是，此处的常数 $a^{(4)}$ 也不同于类光超曲面对应的常数。

If one insists on the necessity that the null and spacelike hypersurface must give the same proportionality constant to the horizon area and take it as a sanity condition of any horizon molecule proposal, then we see that the horizon molecule definition introduced by Barton et al. does not meet this requirement.

如果坚持要求类光和类空超曲面必须给出相同的视界面积比例常数, 并将这一点作为任何视界分子方案的合理性检验条件，那么我们可以看到，Barton 等人引入的视界分子定义不满足该要求。

A final remark about the flat counting with a null hypersurface is that it is free from any IR divergences. This IR divergences could have arisen from points p_- arbitrarily close to the $\sum$ and to the far past of $\mathcal{J}$, e.g., with $v \sim 0, u \to -\infty$ and volume V_f close to zero in the folded plane case, hence not exponentially suppressed. It is not actually difficult to see how this IR divergence is cured within this horizon molecule counting. The requirement that p_- is max-but- n (at least $n = 1$) bounds it away from $\sum$; this is realized in the general integral formula of $< \mathbf{H}_1 >$ by multiplying the exponential term, $e^{-\varrho_c V}$, by the extra volume term $V_{+,f}$, or V_+^n for n -molecule, which has first to vanish before V_f approaches zero. Therefore, the term $V_{+,f}$ accompanying the exponential kills off this IR.

关于类光超曲面下的平直计数的最后一点说明是，它不存在任何红外发散。这类红外发散可能源于任意接近 $\sum$ 且处于 $\mathcal{J}$ 遥远过去的点 p_- ， 例如在折叠平面情形中， $v \sim 0, u \to -\infty$ 和体积 V_f 趋近于零，因此不会得到指数抑制。不难看出，这种红外发散如何在该视界分子计数中得到解决: p_- 需满足极大但非 n (至少对 $n = 1$ 而言) 的要求将其限制在远离 $\sum$ 的区域；这一约束在 $< \mathbf{H}_1 >$ 的一般积分公式中通过给指数项 $e^{-\varrho_c V}$ 乘上额外体积项 $V_{+,f}$ 实现，而对 n 分子则是乘上 V_+^n ，该额外项在 V_f 趋近于零之前必须先变为零。因此，伴随指数项的 $V_{+,f}$ 消除了这个红外发散。

Curved Case

弯曲情形

To investigate the curved case with a null hypersurface the author in [21] took a path similar in spirit to that taken by Barton et al., but now by setting up local Gaussian null coordinates (GNC) adapted to the study of the null hypersurface case.

为了研究零超平面的弯曲情形，文献 [21] 的作者采用了与 Barton 等人本质上相似的思路，但通过构造适用于零超平面情形研究的局部高斯零坐标 (GNC) 完成分析。

For folded null planes, a local coordinate (v, u, y^α) is constructed in a tubular neighborhood $\mathcal{N} \supset \mathcal{J}$. A region $\mathcal{R}_\Lambda$ analogue to $\mathcal{R}_t$ in the spacelike hypersurface case is defined as follows (in time-reversed coordinates):

对于折叠零平面，会在管状邻域 $\mathcal{N} \supset \mathcal{J}$ 中构造局部坐标 (v, u, y^α) 。我们按照如下方式定义了一个类空超平面情形中 $\mathcal{R}_t$ 的类比区域 $\mathcal{R}_\Lambda$ (在时间反演坐标下):

$$\mathcal{R}_\Lambda := \{p \in I^+(\mathcal{J}) \cap \mathcal{N} \mid 0 < v(p) < \Lambda, 0 < u(p) < \Lambda\} \tag{49}$$

where Λ is an intermediate scale between the discreteness and the geometric length of the setting, i.e., $l_c \ll \Lambda \ll L_G$. For the argument used in the spacelike hypersurface case to carry over to the null setting, one has to show that the rescaled expected number of horizon molecules can be reduced to a local integral on $\mathcal{R}_\Lambda$:

其中 Λ 是离散性与该设定几何长度之间的中间尺度，即 $l_c \ll \Lambda \ll L_G$ 。要让类空超平面情形的论证推广到零情形，需要证明缩放后的视界分子期望数可以化简为 $\mathcal{R}_\Lambda$ 上的局部积分:

$$\varrho_c^{\frac{2-d}{d}} < \mathbf{H} >= \varrho_c^{\frac{2-d}{d}+2} \int_{\mathcal{R}_\Lambda} V_+ (p, \lambda) e^{-\varrho_c V(p,\lambda)} dV_p + \cdots \tag{50}$$

where ".....” refer to terms decaying exponentially fast as we go to the limit $l_c \to 0$, and λ is the parameter of the folding hypersurface $\sum'$.

其中“……”代表当我们取极限 $l_c \to 0$ 时指数衰减的项，且 λ 是折叠超曲面 $\sum'$ 的参数。

Based on the discussion of the flat setting, it is not difficult to argue that the above local expansion cannot in general be true. This can be seen by first noticing that although the region (in future-past swapped setting) $v(p) > \Lambda$ poses no problem for all values of $u(p)$, as $\varrho_c V \gg 1$, so that all contributions from this region will be exponentially suppressed in the continuum limit. However, when $u(p) > \Lambda$ this argument fails, and contributions coming from points p close to u -axis, with $v(p) \sim 0$, are not exponentially suppressed, as the volume $V(p)$ is now close to zero for values of u arbitrarily far in the future (or the past in the original setting) of $\mathcal{J}$. Thus, contributions coming from far away along the past light cone of the intersection hypersurface cannot be neglected. It follows then that the above local integral cannot in general count for the dominant contributions to the expected number of horizon molecules in the continuum limit. Therefore, Machet and Wang concluded that this failure is a first indication that the proposal of Barton et al. to count horizon molecules with a null hypersurface is a flawed way to define entropy on causal set setting.

基于对平坦情形的讨论, 不难发现上述局部展开一般并不成立。我们首先可以观察到, 尽管 (未来-过去交换设定下的) 区域 $v(p) > \Lambda$ 对所有 $u(p)$ 的取值都不存在问题，在 $\varrho_c V \gg 1$ 下，该区域的所有贡献在连续极限下都会被指数压制。然而，当 $u(p) > \Lambda$ 时这个论证就不成立了: 来自靠近 u 轴的点 p (满足 $v(p) \sim 0$) 的贡献不会被指数压制，因为对于离 $\mathcal{J}$ 任意远的未来 (原设定中为过去)，体积 $V(p)$ 都趋近于零。因此，相交超曲面过去光锥上远道而来的贡献不能被忽略。由此可得，上述局部积分一般无法统计连续极限下视界分子期望数的主导贡献。因此，Machet 和 Wang 得出结论: 这种失效是首个迹象表明，Barton 等人用零超平面对视界分子计数的提议，是在因果集框架下定义熵的错误方法。

Machet and Wang further argued that a truncated (by hand) local integral in the form (50), in which contributions from the far past of the intersection are excluded, yields a small l_c expansion of the following form for $I^{(d)}(y; \lambda_c, \lambda, \Lambda)$

Machet 和 Wang 进一步指出，对形式为 (50) 的局部积分进行手动截断，排除相交面远过去的贡献后，会为 $I^{(d)}(y; \lambda_c, \lambda, \Lambda)$ 得到如下形式的小 l_c 展开:

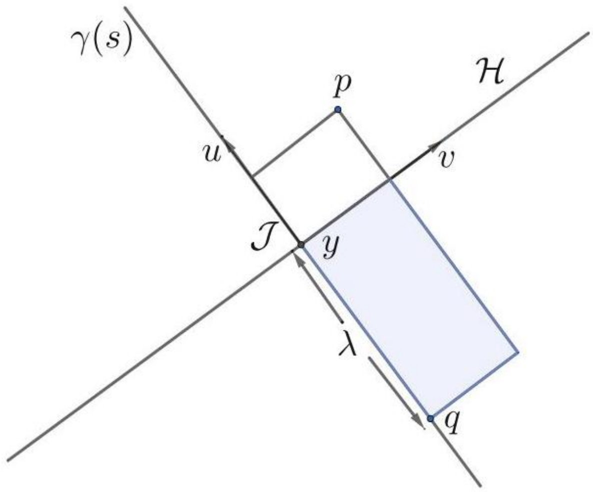
$$I^{(d)}(y; l_c, \lambda, \Lambda) = a^{(d)} + \sum_i c_i^{(d)} \mathcal{F}_i(y, \lambda, \Lambda) + l_c \sum_i b_i^{(d)} \mathcal{G}_i(y, \lambda, \Lambda) + O\left(l_c^2\right)$$

(51)where $a^{(d)}, b_i^{(d)}$, and $c_i^{(d)}$ are dimensionless constants dependent on d . The set $\mathcal{G}_i(y, \lambda, \Lambda)$ is a set of mutually independent geometrical scalars of length L^{-1} evaluated along a geodesic segment. The extra set $\mathcal{F}_i(y, \lambda, \Lambda)$ is a set of independent geometrical dimensionless scalar evaluated along a geodesic segment.

其中 $a^{(d)}, b_i^{(d)}$ 和 $c_i^{(d)}$ 是依赖于 d 的无量纲常数。集合 $\mathcal{G}_i(y, \lambda, \Lambda)$ 是一组沿测地线段计算、相互独立、长度量纲为 L^{-1} 的几何标量。额外集合 $\mathcal{F}_i(y, \lambda, \Lambda)$ 是一组沿测地线段计算、相互独立的无量纲几何标量。

Fig. 11 A sketch of the coordinate system in the relevant skinny diamond setup. V_+ is shaded

图 11 相关细菱形设置下坐标系的示意图。V_+ 为阴影区域



The presence of extra terms $\mathcal{F}_i(y, \lambda, \Lambda)$ carrying geometrical information evaluated along the geodesic segment which, in contrast to the spacelike hypersurface case, survives in the limit $l_c \to 0$ can be given the heuristic explanation based on the above discussion and dimensional analysis.

额外项 $\mathcal{F}_i(y, \lambda, \Lambda)$ 携带沿测地线段计算的几何信息, 且与类空超平面情形不同, 这些项在极限 $l_c \to 0$ 下仍然保留，我们可以结合上述讨论和量纲分析对该现象给出启发式解释。

To further support their claim Machet and Wang worked out an explicit geometrical setting in which $\sum$ is a downward light cone, the past-pointing light cone of a point q which is of affine distance λ away from $\mathcal{J}$. Within the relevant region, in which the V and V_+ tend to a skinny causal interval or diamond, a null Fermi normal coordinate system (b, u, y^α) is set up. Figure 11 gives a sketch of the coordinate system.

为了进一步支撑他们的主张，Machet 和 Wang 构建了一个明确的几何框架，其中 $\sum$ 是下向光锥，即点 q 的过去指向光锥，该点与 $\mathcal{J}$ 的仿射距离为 λ。在相关区域内，V 和 V_+ 趋近于一个细长的因果区间或菱形，该区域建立了零费米正规坐标系 (b, u, y^α)。图 11 给出了该坐标系的示意图。

Both V and V_+ can be expanded around the flat ones and admit the following expansions in $d = 4$:

V 和 V_+ 都可以在平直情形的基础上展开，并可以得到关于 $d = 4$ 的如下展开式:

$$V^{(4)}(u,v,y;\lambda) = V_f^{(4)}(u,v,y;\lambda)\left(1 + F(\lambda,u) + \mathcal{O}\left((u+\lambda)^3, v^3\right)\right) \tag{52}$$

$$V^{(4)}(u,v,y;\lambda) = V_f^{(4)}(u,v,y;\lambda)\left(1 + \widetilde{F}(\lambda,u) + \mathcal{O}\left((u+\lambda)^3, v^3\right)\right) \tag{53}$$

where $F(\lambda,u)$ and $\widetilde{F}(\lambda,u)$ are two dimensionless function involving the integration over the Ricci tensor along the u direction. The assumption here is of course that λ is small relative to the local curvature scales, and u is to be cut off at distance Λ small compared to the local curvature scales.

其中 $F(\lambda,u)$ 和 $\widetilde{F}(\lambda,u)$ 是两个无量纲函数，涉及沿 u 方向对里奇张量的积分。此处的假设当然是 λ 远小于局部曲率尺度，并且 u 在距离 Λ 处截断，而 Λ 本身也远小于局部曲率尺度。

Under the above assumptions along with an extra assumption about the Ricci tensor (an assumption generic enough to support the claim), it was possible to show that $I^{(4)}(y;\lambda_c,\lambda,\Lambda)$ admits the following continuum limit:

结合上述假设，再加上一个关于里奇张量的额外假设 (该假设足够一般，能够支撑相关结论)，可以证明 $I^{(4)}(y;\lambda_c,\lambda,\Lambda)$ 存在如下连续极限:

$$\lim_{l_c\to 0} I^{(4)}(y;l_c,\lambda,\Lambda) = a^{(4)} + \mathcal{R}(y)\, c^{(4)}(\lambda,\Lambda) + \cdots . \tag{54}$$

One can see that the limit is not local to the intersection $\mathcal{J}$, and the area law is distorted. Machet and Wang then concluded that the horizon molecule proposal of Barton et al. does not yield a well-behaved area law when evaluated on a null hypersurface intersecting the horizon.

不难看出，该极限并非局域于交点 $\mathcal{J}$ ，面积定律发生了畸变。因此 Machet 和 Wang 得出结论:Barton 等人的视界分子方案在与视界相交的零超曲面上计算时，无法得到表现良好的面积定律。

Now, whether the above argument is conclusive or just an artifact of the limitation of the expansion adopted by Machet and Wang, which relies on an ad hoc truncation by hand of the integral $I^{(4)}$ and certainly neglecting relevant contributions, remains in our view an unsettled issue. One, for instance, cannot exclude the possibility that in a realistic BH model the area law might be restored. A hint for this possibility is offered by the link counting in 2-dimensional reduced Schwarzschild BH discussed in section ”Horizon Molecules and the Area Law in Two Dimensions”, where the area law is established in the null surface case in a quite subtle way, and at the end the dominant contribution turns out to plainly come from the near horizon links.

在我们看来，上述论证究竟是结论性的，还是仅仅源于 Machet 和 Wang 所采用展开方法的局限性——该方法人为对积分 $I^{(4)}$ 做了特殊截断，必然忽略了相关贡献——这一问题目前仍未有定论。例如，我们不能排除在现实的黑洞模型中面积定律得以恢复的可能性。《二维视界分子与面积定律》一节讨论了二维约化史瓦西黑洞中的链路计数，为这种可能性提供了提示: 在该情形中，面积定律在零曲面情形以相当微妙的方式成立，最终主导贡献显然来自近视界链路。

Discussion and Outlook

讨论与展望

In this survey, we have tried to take the reader through the different attempts to identify the right horizon molecules that would give a good kinematical account for BH entropy within the causal set approach. Despite the fact that there have only been few scattered efforts and practitioners who have devoted their time to this issue, it is undeniable that some progress has been made along different directions. The simplicity, the success in 2-dimensions, and the failure in higher dimensions of the causal link proposal stimulated further investigations and proposals based on triplets and recently have sparked interest in the subject.

在这篇综述中，我们带领读者梳理了各类寻找合适视界分子的研究，这类研究旨在为因果集方法下的黑洞熵给出合理的运动学解释。尽管目前针对该问题的研究和投入其中的研究者都为数不多，但不可否认的是，我们已经在不同方向上取得了一些进展。因果链接方案简洁优美，在二维取得成功却在高维失效，这一情况推动了基于三元组的进一步研究与新方案，近来也重新引发了学界对该领域的兴趣。

The early proposal based on causal links gave some promising results in two dimensions, some of which may seem surprising. Prominent among them is the fact that one finds a universal answer which took the same value in two quite different black hole backgrounds, that of equilibrium and nonequilibrium cases, and that the bulk of the links always reside in the close proximity to the horizon, meaning that the result is controlled by the near horizon geometry. However, a seemingly surprising result is that this value remains finite even in the continuum limit where the fundamental length l_c is sent to zero. In this sense, the replacement of continuous spacetime by a causal set may appear in two dimensions as more of a regularization device than something fundamental. Whether this has any deeper meaning, or whether it might be related to some of the other properties that both quantum field theory and quantum gravity possess in two dimensions [28], that remains an open question. Of note is also the fact that the causal set approach to quantum gravity has been unique in attempting to account for the statistical mechanics of the nonequilibrium horizon.

基于因果链接的早期方案在二维取得了一些令人惊喜的 promising 结果。其中最值得关注的一点是，我们得到了一个普适结果，该结果在平衡态与非平衡态这两种差异极大的黑洞背景中都保持一致，而且绝大多数链接都集中分布在视界附近，说明结果由近邻视界几何主导。然而一个同样令人意外的结果是，即使在基本长度 l_c 趋于零的连续极限下，该结果仍然保持有限。从这个角度看，在二维中用因果集替换连续时空，更像是一种正则化手段，而非根本性的理论变革。这一现象是否存在更深层的含义，或是它是否与二维下量子场论和量子引力共有的其他性质相关 [28]，目前仍是一个开放问题。同样值得注意的是，因果集量子引力方法是独一无二的，它尝试解释非平衡视界的统计力学性质。

It must be added that the above features of the causal link counting are shared by the triplet proposals, i.e., universality of the result and the finiteness in the continuum limit in two dimensions.

需要补充的是，上述因果链接计数的核心特征，包括结果的普适性以及二维连续极限下的有限性，都被三元组方案继承了下来。

Two questions about the links and triplet proposals have so far remained open. The first is to find a way to decide whether the spacelike hypersurface gives the same result as the null one in two dimensions, regardless of the validity of these proposals beyond two dimensions. But as we early mentioned this is a mere mathematical curiosity with little, if any, physical relevance. The second is of importance and concerns the triplet proposal. Although Marr [19] argued that the Λ -triplet would suffer from IR divergences in higher dimensions, the issue is unsettled for l - and z -triplets. Therefore, it would be an interesting exercise to investigate this triplet counting at least in a three-dimensional flat setting. However, as we already mentioned, going beyond two dimensions would make the calculation cumbersome, but if enough time and effort is devoted to this problem, some approximation methods could possibly be devised to extract the leading contributions or settle the divergence issue. Of course, one could also use numerical methods to approach this counting.

> 迄今为止，链接方案和三元组方案都还遗留两个开放问题。第一个问题是，无论这些方案在二维以外是否成立，都需要找到方法验证二维下类空超曲面得到的结果是否和类光超曲面一致。但正如我们此前提到的，这仅仅是一个数学趣味问题，几乎没有物理相关性。第二个问题十分重要，且和三元组方案直接相关。尽管 Marr[19] 指出 Λ 三元组在高维下会存在红外发散，但这一问题对 l 三元组和 z 三元组而言仍未有定论。因此，至少在三维平直背景下研究这类三元组计数会是一项很有意义的工作。不过正如我们已经提到的，突破二维会让计算变得十分繁琐，但如果为该问题投入足够的时间与精力，我们有可能设计出近似方法提取领头阶贡献，或是解决发散问题。当然，我们也可以使用数值方法来进行这类计数。

Unlike the links and the triplet attempts, the Barton et al. proposal has succeeded in giving an expected number of horizon molecules proportional to the area of the horizon intersecting a spacelike hypersurface, for almost all reasonable geometrical settings. However, this proposal has some drawbacks. First, it seems to be inherently discontinuous as one moves from the spacelike hypersurfaces to null ones, giving two different values, i.e., different proportionality constants. Moreover, if one accepts the conclusion of Machet and Wang [21], in a curved geometrical background and for a null hypersurface, the continuum limit of the expected number of Barton et al. horizon molecules is not local to the intersection of the horizon and the hypersurface, yielding to an ill-behaved area law. Nonetheless, if one sticks with spacelike hypersurfaces, the Barton et al. counting provides a good measure for the area of the intersection of the horizon (a null surface) and spacelike hypersurface, which is a promising aspect of such horizon molecule proposal.

> 和链接、三元组这类方案不同，Barton 等人的方案在几乎所有合理几何背景下，都成功得到了符合预期的结果: 视界分子的期望数量正比于视界与类空超曲面的相交面积。但该方案也存在一些缺陷。首先，当我们从类空超曲面过渡到类光超曲面时，该方案本身就存在不连续性，会给出两个不同的结果，也就是不同的比例常数。此外，如果我们接受 Machet 和 Wang[21] 的结论，在弯曲几何背景下的类光超曲面中，Barton 等人视界分子期望数的连续极限并不局域在视界和超曲面的相交处，最终会导致面积定律表现异常。尽管如此，如果我们只考虑类空超曲面，Barton 等人的计数能够很好地度量视界 (一个类光曲面) 和类空超曲面相交面积，这是这类视界分子方案一个很有前景的特点。

Another weakness of Barton et al.'s definition, in our view, is that it lacks the physical intuitive and heuristic picture shared by the links and the triplet (and the diamond) proposals. The elements that underpinned the Barton et al. horizon molecules are points exterior to the black hole and to the past of the hypersurface, with no reference to the future of the hypersurface; hence, it would be hard to view such molecules as heuristically producing entanglement during the course of the causal set growth (or time development).

在我们看来，Barton 等人定义的另一个缺点是，它缺乏链接方案、三元组方案以及菱形方案共有的物理直觉与启发式图景。支撑 Bathing 等人视界分子的要素是黑洞外部、超曲面过去的点，完全没有涉及超曲面未来的部分；因此我们很难从启发式的角度，将这类分子看作因果集生长 (即时间演化) 过程中产生纠缠的单元。

Finally, the common weakness of all proposals discussed in this review, of course, is that they remain at purely kinematical level, and even if a fully successful kinematical identification of the horizon molecules is achieved, no successful proposal can be substantiated or refuted before we possess a fully quantum dynamics of causets. And despite a very important step made by Rideout and Sorkin in developing classical stochastic dynamics for causets, classical sequential growth models [29], building a viable quantum sequential growth dynamics, have so remained challenging [24,30,31].

最后，本综述所讨论的所有方案的共同弱点当然在于，它们目前仍停留在纯运动学层面；即便成功完成了视界分子的运动学识别，在得到完整的因果集量子动力学之前，我们都无法证实或证伪任何合理方案。尽管莱德奥特和索金在构建因果集经典随机动力学方面迈出了重要一步，建立了经典顺序增长模型 [29]，但构建可行的量子顺序增长动力学至今仍极具挑战性 [24,30,31]。

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致谢撰写本综述期间，笔者与费伊·道克和 Ludovico Machet 进行了多次讨论，获益良多；感谢二位解答笔者的问题，并就其工作相关的诸多问题展开讨论。

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